N = 2 Supergravities in Harmonic

N=2 超引力的调和

Superspace

超空间

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Abstract

摘要

Basics of $\mathcal{N} = 2, 4D$ conformal and Einstein supergravities in the harmonic superspace approach are outlined. The crucial merit of this formulation is that the relevant off-shell supermultiplets, in particular $\mathcal{N} = 2, 4D$ superconformal Weyl multiplet, are accommodated by the harmonic-analytic unconstrained prepotentials with a clear geometric meaning, like in the analogous formulation of $\mathcal{N} = 2, 4D$ supersymmetric gauge theory. The fundamental gauge group of conformal supergravity is constituted by the analyticity-preserving diffeo-morphisms of harmonic superspace. The superfield actions of various off-shell versions of $\mathcal{N} = 2$ Einstein supergravity are obtained as the actions of the appropriate harmonic analytic compensators in the background of conformal $\mathcal{N} = 2$ supergravity. The version admitting the most general couplings to quaternion-Kähler matter is the "principal" version with the unconstrained harmonic analytic hypermultiplet superfield as a compensator. It involves an infinite number of auxiliary fields.

本文概述了调和超空间方法中 $\mathcal{N}=2,4D$ 共形爱因斯坦超引力的基础。该表述的核心优点在于，相关的离壳超多重态 (尤其是 $\mathcal{N}=2,4D$ 超共形外尔多重态) 可由具有清晰几何意义的调和解析无约束预势容纳，这和 $\mathcal{N}=2,4D$ 超对称规范理论的类似表述一致。共形超引力的基本规范群由调和超空间保持解析性的微分同胚构成。$\mathcal{N}=2$ 爱因斯坦超引力各类离壳形式的超场作用量，是由共形 $\mathcal{N}=2$ 超引力背景下合适的调和解析补偿场的作用量得到。允许与四元数-Kähler 物质最普遍耦合的版本是“主”版本，它以无约束调和解析超多重态超场作为补偿场，包含无穷多个辅助场。

Keywords

关键词

Supersymmetry - Harmonic superspace - Supergravity

超对称-调和超空间-超引力

In Memory of V.I. Ogievetsky and A.S. Galperin

纪念 V.I. 奥格耶夫斯基与 A.S. 加尔佩林

Introduction

引言

The natural geometric framework for supersymmetric field theories [1-3] is provided by superspace, an extension of Minkowski space (or, of any other bosonic space) by anticommuting fermionic (Grassmann) coordinates [2, 4, 5] (These variables were treated in [2] as fermionic fields given on Minkowski space (Goldstone fermions), while in [4] as independent new anticommuting coordinates.). The fields defined on superspace are called superfields [4, 5]. They naturally describe the supermultiplets of given supersymmetry. The fields forming these supermultiplets come out as coefficients in the expansion of the superfields over Grassmann coordinates. The basic merit of the off-shell superfield approach is the opportunity to formulate the supersymmetric theories in a systematic and consistent way, making manifest their nontrivial intrinsic geometries (e.g., the complex geometry of $\mathcal{N}=1$ supergravity [6]) and their remarkable quantum properties (e.g., the ultraviolet finiteness of $\mathcal{N}=4$ super Yang-Mills theory [7]).

超空间是超对称场论的天然几何框架 [1-3]，它是对闵氏空间 (或任意其他玻色空间) 的扩展，新增了反对易费米子 (格拉斯曼) 坐标 [2, 4, 5] (这些变量在文献 [2] 中被处理为定义在闵氏空间上的费米场 (戈德斯通费米子)，而在文献 [4] 中被处理为独立的新增反对易坐标)。定义在超空间上的场被称为超场 [4,5]，它们可以自然描述给定超对称的超多重态。构成这些超多重态的场正是超场对格拉斯曼坐标展开后的系数。脱壳超场方法的核心优点在于，它能够系统、自洽地表述超对称理论，清晰展现其非平凡的内蕴几何 (例如 $\mathcal{N}=1$ 超引力的复几何 [6]) 与显著的量子性质 (例如 $\mathcal{N}=4$ 超杨-米尔斯理论的紫外有限性 [7])。

The superfield approach to $\mathcal{N} = 1, 4D$ Poincaré supersymmetry was proposed some 50 years ago in the pioneering papers [4, 5, 8] . It took much longer to work out a suitable superfield formalism for extended supersymmetries (i.e., those containing more than one spinor generator). Even in the simplest case of $\mathcal{N} = 2$ supersymmetry, up to 1984, it was unknown how to formulate the relevant theories off shell, in a manifestly supersymmetric form and in terms of unconstrained superfields. The breakthrough came about with the invention of a new type of superspace, the harmonic superspace (HSS) [9-12], as a development of the important concept of Grassmann analyticity [13]. It allowed to construct off-shell unconstrained formulations for all the $\mathcal{N} = 2$ supersymmetric theories ($\mathcal{N} = 2$ matter, Yang-Mills and supergravity) and for $\mathcal{N} = 3$ supersymmetric Yang-Mills theory.

$\mathcal{N} = 1, 4D$ 庞加莱超对称的超场方法早在约 50 年前的开创性论文 [4, 5, 8] 中就被提出。而针对扩展超对称 (即包含多于一个旋量生成元的超对称)，人们花费了更长时间才建立起合适的超场形式体系。即使是最简单的 $\mathcal{N} = 2$ 超对称情形，直到 1984 年，人们仍然不清楚如何将相关理论表述为脱壳、显超对称且用无约束超场描述的形式。随着新型超空间——调和超空间 (HSS)[9-12] 的发明，研究取得了突破，这是格拉斯曼解析性这一重要概念 [13] 的发展成果。它为所有 $\mathcal{N} = 2$ 超对称理论 (物质、杨-米尔斯与超引力 ($\mathcal{N} = 2$) 和 $\mathcal{N} = 3$ 超对称杨-米尔斯理论构造了脱壳无约束表述。

Harmonic $\mathcal{N} = 2$ superspace (HSS) is an extension of the standard $\mathcal{N} = 2$ superspace by the two-dimensional sphere $S^2 \sim SU(2)/U(1)$. In such an extended superspace, a new kind of Grassmann analytic subspace exists, the harmonic analytic one, parametrized by half of the original spinor coordinates [9, 10, 14]. The Grassmann harmonic analyticity is the key to finding the adequate off-shell unconstrained formulations mentioned above, just like $\mathcal{N} = 1$ chirality, the simplest case Grassmann analyticity, forms the basis of the unconstrained superfield formulations of $\mathcal{N} = 1$ supersymmetric theories.

调和 $\mathcal{N} = 2$ 超空间 (HSS) 是标准 $\mathcal{N} = 2$ 超空间经二维球面 $S^2 \sim SU(2)/U(1)$ 扩展得到的空间。在这类扩展超空间中，存在一种新型格拉斯曼解析子空间: 调和解析子空间，由原旋量坐标的一半参数化 [9, 10, 14]。格拉斯曼调和解析性是得到上述合适的脱壳无约束表述的关键，正如 $\mathcal{N} = 1$ 手征性 (最简单的格拉斯曼解析性) 构成了 $\mathcal{N} = 1$ 超对称理论无约束超场表述的基础。

The ultimate goal of the present review paper is to give an account of the basic elements of the HSS formulation of $\mathcal{N} = 2, 4D$ supergravity (SG), though we will also touch some other aspects of the HSS approach.

尽管本文也会涉及调和超空间方法的其他方面，但本综述的最终目标是阐述 $\mathcal{N} = 2, 4D$ 超引力 (SG) 调和超空间表述的基本内容。

Section ”Superspace: Basic Concepts” contains the introductory information about supersymmetry and superspaces. In section ” $\mathcal{N} = 2$ Harmonic Superspace,” the main motivations for HSS are explained and the basic concepts and technical tools of this approach are described. Section ” $\mathcal{N} = 2$ Matter and Gauge Theories in Harmonic Superspace” collects the HSS formulations of $\mathcal{N} = 2$ matter and super Yang-Mills theory as a necessary preparatory step to various versions of $\mathcal{N} = 2$ supergravity. In section ” $\mathcal{N} = 2$ Einstein Supergravity ”From Scratch”,” we ”from scratch” describe the HSS formulation of the simplest (and historically first) Einstein $\mathcal{N} = 2$ supergravity, basically following Ref. [15]. We also explicitly present its linearized superfield action, based on [16] and [17]. In section ”Conformal $\mathcal{N} = 2$ Supergravity,” we describe the HSS formulation of conformal $\mathcal{N} = 2$ SG [18] and then explain, basically on the example of section ” $\mathcal{N} = 2$ Einstein Supergravity ”From Scratch”,” how to pass to various versions of Einstein $\mathcal{N} = 2$ SG by adding the appropriate HSS compensating superfields in the background of conformal SG. We also describe $\mathcal{N} = 2$ supergravity-matter couplings and point out that the most general coupling can be achieved only in the framework of the so-called ”principal” version of $\mathcal{N}=2$SG using the off-shell hypermultiplet as a compensator and so involving an infinite number of auxiliary fields. The concluding section ”Summary and Some Further Problems” gives a brief summary of applications of the HSS approach to supergravities and some related theories, including quite recent applications to the off-shell description of $\mathcal{N}=2$ supersym-metric higher spins [17]. Remarkably, the HSS formulation of the latter theory for arbitrary superspin $\mathbf{s}>2$ reveals a great resemblance with that of $\mathcal{N}=2$SG presented in section ” $\mathcal{N}=2$ Einstein Supergravity ”From Scratch”,” being its rather straightforward generalization. There is also given an (incomplete) list of problems still waiting their resolution within the HSS approach and its proper modifications.

“超空间: 基本概念”小节包含超对称与超空间的入门介绍。在“$\mathcal{N}=2$ 调和超空间”小节中，我们阐释了调和超空间 (HSS) 方法的核心动机，介绍了该方法的基本概念与技术工具。“$\mathcal{N}=2$ 调和超空间中的物质与规范理论”小节整理了 $\mathcal{N}=2$ 物质与超杨-米尔斯理论的 HSS 表述，为研究不同版本的 $\mathcal{N}=2$ 超引力做好必要准备。在“$\mathcal{N}=2$ 从零开始的爱因斯坦超引力”小节中，我们基本遵循文献 [15]，从零开始讲解了最简单 (也是历史上第一个) 爱因斯坦 $\mathcal{N}=2$ 超引力的 HSS 表述。我们还基于文献 [16] 和 [17]，明确给出了它的线性化超场作用量。在“共形 $\mathcal{N}=2$ 超引力”小节中，我们介绍了共形 $\mathcal{N}=2$ 超引力 (SG) 的 HSS 表述 [18]，随后以“$\mathcal{N}=2$ 从零开始的爱因斯坦超引力”小节为例，讲解了如何通过在共形超引力背景中引入合适的 HSS 补偿超场，得到不同版本的爱因斯坦 $\mathcal{N}=2$ SG。我们还介绍了 $\mathcal{N}=2$ 超引力-物质耦合，并指出只有在以脱壳超多重子为补偿器、包含无穷多辅助场的所谓“主版本”$\mathcal{N}=2$SG 框架下，才能得到最一般的耦合形式。最后“总结与待解决问题”小节简要总结了 HSS 方法在超引力及相关理论中的应用，包括其近期在 $\mathcal{N}=2$ 超对称高自旋脱壳描述中的应用 [17]。值得注意的是，任意超自旋 $\mathbf{s}>2$ 对应该理论的 HSS 表述与“$\mathcal{N}=2$ 从零开始的爱因斯坦超引力”小节给出的 $\mathcal{N}=2$SG 表述高度相似，它正是后者直接推广得到的结论。本节也列出了 (部分) 仍待在 HSS 方法及其恰当修改框架内解决的问题。

In this paper, I mainly focus on the HSS approach to $\mathcal{N}=2$ theories and apologize for an inevitable incompleteness of the reference list. The more comprehensive list of references to the component and superfield formulations of $\mathcal{N}=2$ supergravity can be found, e.g., in [19,20].

本文主要聚焦于 $\mathcal{N}=2$ 理论的 HSS 方法，参考文献难免有所疏漏，在此致歉。若需查阅更完整的 $\mathcal{N}=2$ 超引力分量表述与超场表述参考文献，可参见例如 [19,20]。

Superspace: Basic Concepts

超空间: 基本概念

$\mathcal{N}$ -Extended Poincaré Supersymmetry and Superspaces

$\mathcal{N}$ 扩展庞加莱超对称与超空间

The $\mathcal{N}=1$ Poincaré supersymmetry, along with the standard Poincaré group generators $P_m, L_{[m,n]}(m,n=0,1,2,3;P_m$ are the 4-translation generators and $L_{[m,n]}$ the Lorentz group ones), involves the fermionic Weyl generators $Q_\alpha, \bar{Q}_{\dot\alpha} (\alpha, \dot\alpha = 1, 2)$ which transform as $(1/2, 0)$ and $(0, 1/2)$ of the Lorentz group and obey the following anticommutation relations:

$\mathcal{N} = 1$ 庞加莱超对称，搭配标准庞加莱群生成元 ($P_m, L_{[m,n]} (m, n = 0, 1, 2, 3; P_m$ 是 4 个平移生成元，$L_{[m,n]}$ 是洛伦兹群生成元)，包含外尔费米子生成元 $Q_\alpha, \bar{Q}_{\dot\alpha} (\alpha, \dot\alpha = 1, 2)$ ，这些生成元按洛伦兹群的 $(1/2, 0)$ 和 $(0, 1/2)$ 变换，满足如下对易关系:

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2(\sigma^m)_{\alpha\dot\alpha} P_m,\ \{Q_\alpha, Q_\beta\} = \left\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\right\} = 0,\ (\sigma^m)_{\alpha\dot\alpha} = \left(1, \vec{\sigma}\right)_{\alpha\dot\alpha}. \tag{1}$$

$\mathcal{N} > 1$ extended supersymmetry involves N copies of the fermionic generators, each satisfying relations (1)

$\mathcal{N} > 1$ 扩展超对称包含 N 份费米子生成元，每一份都满足关系式 (1)

$$\{Q^i_\alpha, \bar{Q}_{\dot\alpha k}\} = 2\delta^i_k (\sigma^m)_{\alpha\dot\alpha} P_m,\ \left\{Q^i_\alpha, Q^k_\beta\right\} = \left\{\bar{Q}_{\dot\alpha i}, \bar{Q}_{\dot\beta k}\right\} = 0. \tag{2}$$

Here $i = 1, \ldots N$ is the index of the fundamental representation of the internal automorphism symmetry (or R-symmetry) group $U(N)$ (Some important theories, e.g., $\mathcal{N} = 4$ super Yang-Mills theory, in fact respect only $SU(N)$ R-symmetry.).

此处 $i = 1, \ldots N$ 是内自同构对称群 (即 R 对称群) $U(N)$ 基础表示的指标 (部分重要理论，例如 $\mathcal{N} = 4$ 超杨-米尔斯理论，实际上仅满足 $SU(N)$ R 对称性)。

The natural way to realize $\mathcal{N}$ -extended Poincaré supersymmetry is to use the standard superspace [4, 5]

实现 $\mathcal{N}$ 扩展庞加莱超对称的自然方式是使用标准超空间 [4, 5]

$$\mathbb{R}^{4|4N} = \left(x^a, \theta^\alpha_i, \bar{\theta}^{\dot\alpha i}\right),\ i = 1, 2, \ldots, N \tag{3}$$

involving the spinor anticommuting coordinates $\theta^\alpha_i, \bar{\theta}^{\dot\alpha i}$, in addition to the commuting x^a . Their transformation rules under the Poincaré group are evident, while the transformations under supersymmetry (supertranslations with anticommuting parameters $\varepsilon^\alpha_i, \bar{\varepsilon}^{\dot\alpha i}$) are given by

除了对易的 x^a 之外，还包含旋量反对易坐标 $\theta^\alpha_i, \bar{\theta}^{\dot\alpha i}$ 。它们在庞加莱群下的变换规则是显然的，而在超对称下 (带反对易参数 $\varepsilon^\alpha_i, \bar{\varepsilon}^{\dot\alpha i}$ 的超平移) 的变换由下式给出

$$\delta x^a = i\left(\varepsilon^i \sigma^a \bar{\theta}_i - \theta^i \sigma^a \bar{\varepsilon}_i\right),\ \delta\theta^\alpha_i = \varepsilon^\alpha_i,\ \delta\bar{\theta}^{\dot\alpha i} = \bar{\varepsilon}^{\dot\alpha i}. \tag{4}$$

Superfields $\Phi\left(x, \theta, \bar{\theta}\right)$ are defined as functions on this superspace and their transformation law is completely determined by the superalgebra (2). For example, for a scalar superfield

超场 $\Phi\left(x, \theta, \bar{\theta}\right)$ 被定义为该超空间上的函数，其变换规律完全由超代数 (2) 确定。例如，对于标量超场

$$\Phi'\left(x',\theta',\overline{\theta}'\right)=\Phi\left(x,\theta,\overline{\theta}\right). \tag{5}$$

This law is model-independent. Expanding $\Phi\left(x,\theta,\overline{\theta}\right)$ in powers of the spinor (anticommuting, hence nilpotent) variables $\theta,\overline{\theta}$ produces a finite set of ordinary component fields $f(x),\psi^{\alpha}(x),\ldots$

该规律与模型无关。将 $\Phi\left(x,\theta,\overline{\theta}\right)$ 按旋量 (反对易，因此幂零) 变量 $\theta,\overline{\theta}$ 作幂级数展开，会得到有限个普通分量场 $f(x),\psi^{\alpha}(x),\ldots$

$\mathcal{N}$ -extended supersymmetry can also be realized in the chiral superspace $\mathbb{C}^{4|2N}$ which is complex and involves only half of the spinor coordinates:

$\mathcal{N}$ 扩展超对称也可以在手征超空间 $\mathbb{C}^{4|2N}$ 中实现，手征超空间是复空间，仅包含一半的旋量坐标:

$$\delta x_L^a=-2i\theta_L^i\sigma^a\overline{\varepsilon}_i,\ \delta\theta_{Li}^{\alpha}=\varepsilon_i^{\alpha}. \tag{6}$$

The real superspace $\mathbb{R}^{4|4N}$ forms a real hypersurface in the complex superspace $\mathbb{C}^{4|2N}$:

实超空间 $\mathbb{R}^{4|4N}$ 是复超空间 $\mathbb{C}^{4|2N}$ 中的一个实超曲面:

$$x_L^a=x^a+i\theta^i\sigma^a\overline{\theta}_i,\ \theta_{Li}^{\alpha}=\theta_i^{\alpha}. \tag{7}$$

The chiral superfields [8] $\Phi(x_L,\theta_L)=\Phi\left(x+i\theta\sigma\overline{\theta},\theta\right)$ defined in $\mathbb{C}^{4|2N}$ can be viewed as Grassmann analytic superfields. Indeed, they obey the constraint

定义在 $\mathbb{C}^{4|2N}$ 中的手征超场 [8] $\Phi(x_L,\theta_L)=\Phi\left(x+i\theta\sigma\overline{\theta},\theta\right)$ 可以被视为格拉斯曼解析超场。事实上，它们满足约束

$$\bar{D}_{\dot{\alpha}i}\Phi=0, \tag{8}$$

where $\bar{D}_{\dot{\alpha}i}$ is the covariant (i.e., commuting with the supersymmetry transformations) spinor derivative

其中 $\bar{D}_{\dot{\alpha}i}$ 是协变旋量导数 (即与超对称变换对易的导数)

$$\bar{D}_{\dot{\alpha}i}=-\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}i}}-i(\theta_i\sigma^a)_{\dot{\alpha}}\frac{\partial}{\partial x^a}. \tag{9}$$

Together with

连同

$$D_{\alpha}^i=\frac{\partial}{\partial\theta_i^{\alpha}}+i\left(\sigma^a\overline{\theta}^i\right)_{\alpha}\frac{\partial}{\partial x^a} \tag{10}$$

they form an algebra similar to (2)

它们构成一个类似 (2) 的代数

$$\{D^i_\alpha, \bar{D}_{\dot{\alpha}k}\} = -2i\delta^i_k(\sigma^a)_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^a}. \tag{11}$$

In the basis $\left(x_L, \theta_L, \overline{\theta}\right)$ the derivative $\bar{D}_{\dot{\alpha}i}$ takes the "short" form $\bar{D}_{\dot{\alpha}i} = -\partial/\partial\overline{\theta}^{\dot{\alpha}i}$. Then the constraint (8) becomes a sort of Grassmann Cauchy-Riemann condition

在基底 $\left(x_L, \theta_L, \overline{\theta}\right)$ 下，导数 $\bar{D}_{\dot{\alpha}i}$ 呈现“短”形式 $\bar{D}_{\dot{\alpha}i} = -\partial/\partial\overline{\theta}^{\dot{\alpha}i}$，此时约束条件 (8) 变为一种格拉斯曼柯西-黎曼条件

$$\frac{\partial\Phi}{\partial\overline{\theta}^{\dot{\alpha}}_i} = 0 \tag{12}$$

which implies that Φ is a function of θ_L but not of $\overline{\theta}$ (cf. the standard Cauchy-Riemann condition $\partial/\partial\bar{z}f(z) = 0$, which means that the function f depends on the variable z and not on its complex conjugate $\bar{z}$). This simple version of Grassmann analyticity [13] works effectively in $\mathcal{N} = 1$ supersymmetry.

这意味着 Φ 是 θ_L 的函数，而非 $\overline{\theta}$ 的函数 (对比标准柯西-黎曼条件 $\partial/\partial\bar{z}f(z) = 0$，该条件说明函数 f 仅依赖变量 z，不依赖其复共轭 $\bar{z}$)。这种简单形式的格拉斯曼解析性 [13] 在 $\mathcal{N} = 1$ 超对称中能有效发挥作用。

The important concept of Grassmann analyticity admits nontrivial generalizations, which underlie the $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supersymmetric theories, and these generalized Grassmann analyticities constitute the basis of the harmonic superspace approach.

格拉斯曼解析性这一重要概念可进行非平凡推广，这些推广是 $\mathcal{N} = 2$ 和 $\mathcal{N} = 3$ 超对称理论的基础，而这些广义格拉斯曼解析性构成了调和超空间方法的核心。

In general, the fields appearing in the θ -expansion of a superfield form reducible supermultiplets. To single out the irreducible multiplets, one should subject the carrier superfield to certain manifestly supersymmetric constraints and/or admit some gauge freedom for it.

一般而言，超场 θ 展开中出现的场构成可约超多重态。为分离出不可约多重态，需要对承载超场施加某些明显超对称的约束，和/或为其允许一定的规范自由度。

It should by emphasized that finding the adequate superspace for a given theory is, as a rule, a nontrivial problem. The superspaces $\mathbb{R}^{4|4N}$ and $\mathbb{C}^{4|2N}$ prove to be appropriate for off-shell formulations only in the simplest case of $\mathcal{N} = 1$ supersymmetry. These "standard" superspaces are not so useful in the extended ($\mathcal{N} > 1$) supersymmetric theories.

需要强调的是，给定量论找到合适的超空间通常是一个非平凡问题。只有在 $\mathcal{N} = 1$ 超对称的最简单情形下，超空间 $\mathbb{R}^{4|4N}$ 和 $\mathbb{C}^{4|2N}$ 才适用于离壳 formulation。这些“标准”超空间在推广的 ($\mathcal{N} > 1$) 超对称理论中并没有多大用处。

Chirality as a Key to $\mathcal{N}=1$ Theories

手征性是 $\mathcal{N}=1$ 理论的关键

The chiral ($\mathcal{N}=1$ analytic) superspace $\mathbb{C}^{4|2}$ forms the basis of all $\mathcal{N}=1$ theories: they are either formulated in terms of chiral superfields (matter and its self-couplings) or follow from gauge principles, which respect chirality (super Yang-Mills (SYM) and supergravity (SG) theories and their couplings to matter).

手征 ($\mathcal{N}=1$ 解析超空间 $\mathbb{C}^{4|2}$ 构成了所有 $\mathcal{N}=1$ 理论的基础: 这些理论要么基于手征超场 (物质及其自耦合) 构建，要么遵循满足手征性要求的规范原理 (超杨-米尔斯 (SYM)、超引力 (SG) 理论及其与物质的耦合都符合这一点)。

The most general action of $\mathcal{N}=1$ matter is the action of n chiral superfields $\Phi_A(x_L,\theta),(A=1,\dots n)$, and it is given by

$\mathcal{N}=1$ 物质的最一般作用量就是 n 个手征超场 $\Phi_A(x_L,\theta),(A=1,\dots n)$ 的作用量，其形式为

$$S_\Phi=\int d^4xd^4\theta K\left(\Phi_A,\overline{\Phi}_B\right)+\left[\int d^4x_L V\left(\Phi_A\right)+\text{ c.c}\right]. \tag{13}$$

In components, the first term gives a sigma model-type action, with the most general $2n$ -dimensional Kähler target metric for which $K\left(\phi,\overline{\phi}\right)$ is the Kähler potential [21]. The second term, after elimination of the auxiliary fields by their equations of motion, yields the most general scalar potential of $\phi,\overline{\phi}$ consistent with $\mathcal{N}=1$ supersymmetry, plus the appropriate Yukawa couplings of physical fermionic fields. Any other off-shell matter representation of $\mathcal{N}=1$ supersymmetry (e.g., the so-called tensor $\mathcal{N}=1$ multiplet) is described by the properly constrained $\mathcal{N}=1$ superfields related to the chiral ones via the appropriate duality transformation [22].

分解为分量形式后，第一项给出 sigma 模型型作用量，对应最一般的 $2n$ 维凯勒目标度量，其中 $K\left(\phi,\overline{\phi}\right)$ 是凯勒势 [21]。第二项通过运动方程消去辅助场后，得到满足 $\phi,\overline{\phi}$ 、与 $\mathcal{N}=1$ 超对称相容的最一般标量势，以及物理费米子场对应的汤川耦合。$\mathcal{N}=1$ 超对称的所有其他离壳物质表示 (例如所谓的张量 $\mathcal{N}=1$ 多重态) 都可通过适当约束的 $\mathcal{N}=1$ 超场描述，这类超场可通过恰当的对偶变换与手征超场联系起来 [22]。

The fundamental object (prepotential) of $\mathcal{N}=1$ SYM theory carrying the irreducible field content of the off-shell $\mathcal{N}=1$ vector multiplet (gauge field $b_m(x)$, gaugino $\psi_\alpha(x),\overline{\psi}_{\dot\alpha}(x)$ and the auxiliary field $D(x)$, all taking values in the adjoint representation of some gauge group) is the real scalar superfield $V\left(x,\theta,\overline{\theta}\right)$ [23]. Its gauge transformation, up to nonlinear terms, is given by

$\mathcal{N}=1$ 超杨-米尔斯理论中承载离壳 $\mathcal{N}=1$ 矢量多重态不可约场内容 (规范场 $b_m(x)$ 、戈金诺 $\psi_\alpha(x),\overline{\psi}_{\dot\alpha}(x)$ 和辅助场 $D(x)$ ，所有场都属于某个规范群的伴随表示) 的基本对象 (前势) 是实标量超场 $V\left(x,\theta,\overline{\theta}\right)$ [23]。忽略非线性项后，其规范变换可写为

$$V'\left(x,\theta,\overline{\theta}\right)=V\left(x,\theta,\overline{\theta}\right)+\frac{i}{2}\left(\Lambda\left(x_L,\theta\right)-\overline{\Lambda}\left(x_R,\overline{\theta}\right)\right)+\mathcal{O}\left(V\right), \tag{14}$$

where Λ and $\overline{\Lambda}$ are conjugate gauge-algebra valued superfield parameters, defined as unconstrained functions on the left- and right-handed $\mathcal{N}=1$ chiral subspaces. The maximally reduced form of $V\left(x,\theta,\overline{\theta}\right)$ (Wess-Zumino gauge) is as follows

其中 Λ 和 $\overline{\Lambda}$ 是共轭的、取值于规范代数的超场参数，定义为左手和右手 $\mathcal{N}=1$ 手征子空间上的无约束函数。$V\left(x,\theta,\overline{\theta}\right)$ 的最大约化形式 (韦斯-祖米诺规范) 如下

$$V\left(x,\theta,\overline{\theta}\right)=\theta\sigma^{n}\overline{\theta}b_{n}+\left(\overline{\theta}\right)^{2}\theta^{\alpha}\psi_{\alpha}+(\theta)^{2}\overline{\theta}_{\dot{\alpha}}\overline{\psi}^{\dot{\alpha}}+(\theta)^{2}\left(\overline{\theta}\right)^{2}D, \tag{15}$$

$$\delta A_{n}=\partial_{n}\lambda_{0},\ \lambda_{0}\equiv-\frac{1}{2}\left(\Lambda+\overline{\Lambda}\right)\Big|_{\theta=\overline{\theta}=0}.$$

The fields in (15) are recognized as the irreducible off-shell $\mathcal{N}=1$ vector multiplet. From (14), it follows that the fundamental gauge group of $\mathcal{N}=1$ SYM theory is represented by chiral superfield gauge parameters. The differential geometry constraints defining this theory in the superspace $\mathbb{R}^{4|4}$ were given in [24]

式 (15) 中的场就是不可约离壳 $\mathcal{N}=1$ 矢量多重态。由式 (14) 可知，$\mathcal{N}=1$ 超杨-米尔斯理论的基本规范群由手征超场规范参数表示。定义该理论在超空间 $\mathbb{R}^{4|4}$ 中的微分几何约束已在文献 [24] 中给出

$$\left\{\mathcal{D}_{\alpha},\mathcal{D}_{\beta}\right\}=\left\{\overline{\mathcal{D}}_{\dot{\alpha}},\overline{\mathcal{D}}_{\dot{\beta}}\right\}=0. \tag{16}$$

Here $\mathcal{D}_{\alpha}=D_{\alpha}+iA_{\alpha}$ is a gauge-covariantized spinor derivative. These constraints are just the integrability conditions for the existence of chiral $\mathcal{N}=1$ superfields in the full interacting case, thus expressing the fact that $\mathcal{N}=1$ SYM theory is fully determined by the requirement of preservation of $\mathcal{N}=1$ chirality, the simplest form of Grassmann analyticity.

此处 $\mathcal{D}_{\alpha}=D_{\alpha}+iA_{\alpha}$ 是规范协变旋量导数。这些约束恰好就是全相互作用情形下手征 $\mathcal{N}=1$ 超场存在的可积性条件，因此说明 $\mathcal{N}=1$ 超杨-米尔斯理论完全由保持 $\mathcal{N}=1$ 手征性——格拉斯曼解析性最简单的形式——这一要求确定。

Finally, the superspace geometry of $\mathcal{N}=1$SG [25] is also fully fixed by the chirality-preservation principle.

最后，$\mathcal{N}=1$SG 的超空间几何 [25] 也完全由手征性保持原理固定。

The underlying gauge group of conformal $\mathcal{N}=1$SG is just the group of general diffeomorphisms of the chiral superspace [6]:

共形 $\mathcal{N}=1$SG 的底层规范群正是手征超空间的一般微分同胚群 [6]:

$$\delta x_{L}^{m}=\lambda^{m}\left(x_{L},\theta_{L}\right),\ \delta\theta_{L}^{\mu}=\lambda^{\mu}\left(x_{L},\theta_{L}\right), \tag{17}$$

with $\lambda^{m},\lambda^{\mu}$ being arbitrary complex functions of their arguments. The basic gauge prepotential of conformal $\mathcal{N}=1$SG is an axial-vector superfield $H^{m}\left(x,\theta,\overline{\theta}\right)$ appearing as the imaginary part of the bosonic chiral coordinate,

其中 λ^m, λ^μ 是关于其自变量的任意复函数。共形 $\mathcal{N} = 1$SG 的基本规范预势是轴矢量超场 $H^m\left(x, \theta, \bar{\theta}\right)$，它表现为玻色手征坐标的虚部,

$$x_L^m = x^m + iH^m\left(x, \theta, \bar{\theta}\right),\ \theta^\mu = \theta_L^\mu,\ \bar{\theta}^{\dot{\mu}} = \overline{(\theta^\mu)}. \tag{18}$$

It possesses a nice geometric meaning: it specifies the superembedding of real $\mathcal{N} = 1$ superspace $\left(x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}}\right)$ as a hypersurface into the complex chiral $\mathcal{N} = 1$ superspace $\left(x_L^m, \theta_L^\mu\right)$ (It was shown in [26] that the geometric meaning of the $\mathcal{N} = 1$ SYM prepotential $V\left(x, \theta, \bar{\theta}\right)$ is to some extent similar to that of $H^m\left(x, \theta, \bar{\theta}\right)$. The superfield V also specifies a real $(4 \mid 4)$ dimensional hypersurface, this time in the product of $\mathcal{N} = 1$ chiral superspace and the internal coset space G^c/G , where G^c is the complexification of the gauge group G .). Through the relations (18), the transformations (17) generate field-dependent nonlinear transformations of the $\mathcal{N} = 1$ superspace coordinates $\left(x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}}\right)$ and of the superfield $H^m\left(x, \theta, \bar{\theta}\right)$. The field content of H^m can be revealed in the WZ gauge:

它具有清晰的几何意义: 它指明了实 $\mathcal{N} = 1$ 超空间 $\left(x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}}\right)$ 作为超曲面嵌入到复手征 $\mathcal{N} = 1$ 超空间 $\left(x_L^m, \theta_L^\mu\right)$ 中的超嵌入 (文献 [26] 已证明，$\mathcal{N} = 1$ SYM 预势 $V\left(x, \theta, \bar{\theta}\right)$ 的几何意义在一定程度上与 $H^m\left(x, \theta, \bar{\theta}\right)$ 相似。超场 V 同样指明了一个实 $(4 \mid 4)$ 维超曲面，该超曲面位于 $\mathcal{N} = 1$ 手征超空间与内陪集空间 G^c/G 的乘积中，其中 G^c 是规范群 G 的复化。)。通过关系式 (18)，变换 (17) 生成了 $\mathcal{N} = 1$ 超空间坐标 $\left(x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}}\right)$ 和超场 $H^m\left(x, \theta, \bar{\theta}\right)$ 的依赖场的非线性变换。H^m 的场内容可在 WZ 规范中显现:

$$H_{WZ}^m = \theta\sigma^a\bar{\theta}e_a^m + \left(\bar{\theta}\right)^2\theta^\mu\psi_\mu^m + (\theta)^2\bar{\theta}_{\dot{\mu}}\bar{\psi}^{m\dot{\mu}} + (\theta)^2\left(\bar{\theta}\right)^2A^m. \tag{19}$$

Here one finds the vierbein e_a^m representing the conformal graviton (gauge-independent spin 2 off-shell), the gravitino ψ_μ^m (spins 3/2), and the gauge field A^m (spin 1) of the local γ_5 R-symmetry, just $(8+8)$ off-shell degrees of freedom forming $\mathcal{N} = 1$ Weyl multiplet. Various versions of $\mathcal{N} = 1$ Einstein SG actions can be constructed as the actions of various $\mathcal{N} = 1$ superfield compensators in the background of $\mathcal{N} = 1$ Weyl multiplet. All of these actions are related to the version with a chiral compensator via appropriate duality transformations. Once again, the basic differential supergeometry constraints of $\mathcal{N} = 1$SG have the interpretation of integrability conditions for the existence of $\mathcal{N} = 1$ chiral superfields in the full curved case.

此处我们可以得到表示共形引力子的标架场 e_a^m (离壳、规范无关自旋 2)、引力微子 ψ_μ^m (自旋 3/2)，以及局部 γ_5 R 对称性的规范场 A^m (自旋 1)，恰好有 $(8+8)$ 个离壳自由度，构成 $\mathcal{N} = 1$ 外尔多重态。多种形式的 $\mathcal{N} = 1$ 爱因斯坦超引力作用量都可以通过 $\mathcal{N} = 1$ 外尔多重态背景下不同 $\mathcal{N} = 1$ 超场补偿子的作用量构造得到。所有这些作用量都可以通过适当的对偶变换与手征补偿子的形式联系起来。此外，在全弯曲情形下，$\mathcal{N} = 1$SG 的基本微分超几何约束可以解释为 $\mathcal{N} = 1$ 手征超场存在性的可积性条件。

In theories with extended $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supersymmetries, this remarkable chirality-preservation principle is substituted by the principle of preservation of a different type of Grassmann analyticity, the Grassmann harmonic one.

在具有扩展 $\mathcal{N}=2$ 和 $\mathcal{N}=3$ 超对称的理论中，这一显著的手征守恒原理被另一种不同类型的格拉斯曼解析性——格拉斯曼调和解析性的守恒原理所取代。

$\mathcal{N}=2$ Harmonic Superspace

$\mathcal{N}=2$ 调和超空间

Why Standard Superspaces Are Not Enough for $\mathcal{N}=2$ Case

为什么标准超空间不足以描述 $\mathcal{N}=2$ 情形

In the framework of the standard superspaces $\mathbb{R}^{4|8}$ and $\mathbb{C}^{4|4}$ it proved impossible to find an off-shell action principle for an unconstrained description of all $\mathcal{N}=2$ supersymmetric theories.

在标准超空间 $\mathbb{R}^{4|8}$ 和 $\mathbb{C}^{4|4}$ 的框架下，我们无法找到一种脱壳作用量原理，来无约束地描述所有 $\mathcal{N}=2$ 超对称理论。

The basic problem with extended superspace $\left(x^m, \theta_i^\alpha, \overline{\theta}^{\dot{\alpha}i}\right)$ was that the corresponding superfields, due to the large number of Grassmann coordinates, contain too many irreducible supermultiplets. So they should be either strongly constrained or subjected to gauge transformations with an a priori unclear geometric meaning. Another problem was that some constraints imply the equations of motion for the fields involved, which makes impossible to find an invariant off-shell action for them. For instance, in the $\mathcal{N}=2$ case, the simplest matter multiplet (the analog of $\mathcal{N}=1$ chiral multiplet) is the hypermultiplet [27, 28] , which is represented by a complex $SU(2)$ doublet superfield $q^i\left(x,\theta,\overline{\theta}\right)(i=1,2)$ subjected to the constraints

扩展超空间 $\left(x^m, \theta_i^\alpha, \overline{\theta}^{\dot{\alpha}i}\right)$ 的核心问题在于，由于格拉斯曼坐标数量较多，对应的超场包含了过多的不可约超多重态。因此这类超场要么需要施加强约束，要么需要遵守先验几何意义不明确的规范变换。另一个问题是，部分约束会直接给出涉及场的运动方程，导致我们无法找到这类场的不变脱壳作用量。例如，在 $\mathcal{N}=2$ 情形中，最简单的物质多重态 ($\mathcal{N}=1$ 手征多重态的类比) 是超多重态 [27, 28] ，它由满足如下约束的复 $SU(2)$ 二重态超场 $q^i\left(x,\theta,\overline{\theta}\right)(i=1,2)$ 描述:

$$D_\alpha^{(i}q^{k)} = \bar{D}_{\dot{\alpha}}^{(i}q^{k)} = 0. \tag{20}$$

Here $D_\alpha^i, \bar{D}_{\dot{\alpha}}^k$ are $\mathcal{N}=2$ spinor covariant derivatives satisfying the relations (11). On shell this supermultiplet contains four scalar fields forming an $SU(2)_A$ doublet $f^i(x)$ and two isosinglet spinor fields $\psi^\alpha(x), \overline{\kappa}^{\dot{\alpha}}(x)$. Using (11), it is a direct exercise to check that (20) gives rise to the equations of motion for the physical component fields in $q^i = f^i + \theta^{i\alpha}\psi_\alpha + \overline{\theta}^i_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} + \ldots$, viz.,

此处 $D_\alpha^i, \bar{D}_{\dot{\alpha}}^k$ 是满足关系式 (11) 的 $\mathcal{N}=2$ 旋量协变导数。壳上该超多重态包含四个标量场，它们构成一个 $SU(2)_A$ 二重态 $f^i(x)$ ，以及两个同标量旋量场 $\psi^\alpha(x), \overline{\kappa}^{\dot{\alpha}}(x)$ 。利用 (11) 可以直接验证，(20) 给出了 $q^i = f^i + \theta^{i\alpha}\psi_\alpha + \overline{\theta}^i_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}} + \ldots$ 中物理分量场的运动方程，即:

$$\Box f^i = 0,\ \partial_m \psi \sigma^m = \sigma^m \partial_m \overline{\chi} = 0. \tag{21}$$

This phenomenon is a reflection of the ”no-go” theorem [29, 30] stating that no off-shell representation for a hypermultiplet in its ”complex form” (i.e., with the bosonic fields arranged in an $SU(2)$ doublet) can be achieved with any finite number of auxiliary fields. At the time, it was not clear if there existed a way to circumvent this theorem and to write some kind of an off-shell action for the hypermultiplet.

这一现象反映了“不可行”定理 [29,30]: 不存在带有限个辅助场的“复形式”超多重态 (即玻色场构成 $SU(2)$ 二重态的超多重态) 的脱壳表示。在当时，人们并不清楚是否存在方法绕过该定理，为超多重态构造某种脱壳作用量。

A further problem was the lack of a geometric unconstrained formulation of $\mathcal{N} = 2$ SYM theory, similar to the prepotential formulation of $\mathcal{N} = 1$ SYM. The differential geometry constraints defining this theory were given in [24]

更进一步的问题是，缺乏 $\mathcal{N} = 2$ 超对称杨-米尔斯理论的几何无约束表述，无法类比 $\mathcal{N} = 1$ 超对称杨-米尔斯的预势表述。定义该理论的微分几何约束已在文献 [24] 中给出:

$$\left\{\mathcal{D}_{\alpha}^{(i)}, \mathcal{D}_{\beta}^{k)}\right\} = \left\{\overline{\mathcal{D}}_{\dot{\alpha}}^{(k}, \overline{\mathcal{D}}_{\dot{\beta}}^{i)}\right\} = \left\{\mathcal{D}_{\alpha}^{(i}, \overline{\mathcal{D}}_{\dot{\beta}}^{k)}\right\} = 0, \tag{22}$$

where $\mathcal{D}_{\alpha}^{i} = D_{\alpha}^{i} + i\mathcal{A}_{\alpha}^{i}$ is a gauge-covariantized spinor derivative. Mezincescu was the first to find the solution of these constraints in the Abelian case through an unconstrained prepotential [31]. However, the latter possesses a non-standard dimension -2, and the corresponding gauge freedom does not admit a geometric interpretation. It was unclear whether something like the nice geometric interpretation of the $\mathcal{N} = 1$ SYM gauge group and prepotential V could be found in the $\mathcal{N} = 2$ (and $\mathcal{N} > 2$) case. The same problem existed for $\mathcal{N} = 2$ superfield SG.

其中 $\mathcal{D}_{\alpha}^{i} = D_{\alpha}^{i} + i\mathcal{A}_{\alpha}^{i}$ 是规范协变旋量导数。梅津切斯库是第一个在阿贝尔情形下通过无约束预势找到这些约束解的学者 [31]，但该预势具有非标准维度-2，且对应的规范自由度不具备几何解释。当时人们并不清楚，能否在 $\mathcal{N} = 2$ (及 $\mathcal{N} > 2$) 情形中得到类似 $\mathcal{N} = 1$ 超对称杨-米尔斯规范群和预势 V 的清晰几何诠释。同样的问题也存在于 $\mathcal{N} = 2$ 超场超引力中。

In [13] Galperin, Ivanov and Ogievetsky observed that extended supersymmetry, besides standard chiral superspaces generalizing the $\mathcal{N} = 1$ one, also admits some other type of invariant subspaces, which were called ”Grassmann-analytic.” Like in the case of chiral superspaces, these subspaces are obtained by passing to some new basis in the general superspace, such that the spinor covariant derivatives with respect to a subset of the Grassmann variables become ”short” in it. Then one can impose Grassmann Cauchy-Riemann conditions with respect to these variables, while preserving the full extended supersymmetry. In the simplest $\mathcal{N} = 2$ case, allowing the $U(2)$ automorphism symmetry to be broken down to $O(2)$, and making the appropriate shift of x^m , one can define the complex ” $O(2)$ analytic subspace”

在文献 [13] 中，加尔佩林、伊万诺夫和奥吉耶夫斯基指出，除了推广 $\mathcal{N}=1$ 的标准手征超空间外，扩展超对称还允许存在另一种不变子空间，这类子空间被称为“格拉斯曼解析子空间”。与手征超空间的情况类似，这些子空间是通过转换到一般超空间的一组新基得到的，在该组基下，对一部分格拉斯曼变量的旋量协变导数变得“短”。随后我们可以对这些变量施加格拉斯曼柯西-黎曼条件，同时保留完整的扩展超对称。在最简单的 $\mathcal{N}=2$ 情形下，允许 $U(2)$ 自同构对称性破缺到 $O(2)$，并对 x^m 做适当平移后，就可以定义复的“$O(2)$ 解析子空间”

$$\left(\tilde{x}^m, \theta^1_\alpha + i\theta^2_\alpha, \bar{\theta}^1_{\dot{\alpha}} + i\bar{\theta}^2_{\dot{\alpha}}\right), \tag{23}$$

which is closed under $\mathcal{N}=2$ supersymmetry, and the related Grassmann-analytic superfields. It was natural to assume that this new type of analyticity plays a fundamental role in extended supersymmetry, similar to chirality in the $\mathcal{N}=1$ case. In [32] it was found that the hypermultiplet constraints (20) imply that different components of $\mathcal{N}=2$ superfield q^i ”live” on different $O(2)$ -analytic subspaces. Since (20) is $SU(2)$ covariant, it was tempting to ” $SU(2)$ -covariantize” the $O(2)$ analyticity.

它在 $\mathcal{N}=2$ 超对称下封闭，还对应关联的格拉斯曼解析超场。我们很自然可以认为，这类新型解析性在扩展超对称中发挥着基础性作用，就像手征性在 $\mathcal{N}=1$ 情形中一样。文献 [32] 发现，超多重子约束 (20) 表明 $\mathcal{N}=2$ 超场 q^i 的不同分量“位于”不同的 $O(2)$ 解析子空间上。由于 (20) 是 $SU(2)$ 协变的，人们很自然想要对 $O(2)$ 解析性做“$SU(2)$ 协形变”。

All these problems were solved with the invention of harmonic superspace [9-12,14]. The papers [6, 13, 26] and [32] turned out to be the basic benchmarks on the way toward this new concept.

所有这些问题都随着调和超空间的提出 [9-12,14] 得到解决。文章 [6, 13, 26] 和文献 [32] 正是通向这一新概念道路上的核心里程碑。

Harmonic Superspace: The Definition

调和超空间: 定义

$\mathcal{N}=2$ harmonic superspace (HSS) $\mathbb{H}^{4+2|8}$ is defined as the product

$\mathcal{N}=2$ 调和超空间 (HSS) $\mathbb{H}^{4+2|8}$ 被定义为直积

$$\mathbb{H}^{4+2|8} = \mathbb{R}^{4|8} \otimes S^2 = \left(x^m, \theta_{\alpha i}, \bar{\theta}^k_{\dot{\beta}}\right) \otimes \left(u^+_i, u^-_k\right). \tag{24}$$

Here $S^2 \sim SU(2)_A/U(1)$, with $SU(2)_A$ being the automorphism group of the $\mathcal{N}=2$ superalgebra. The internal 2-sphere S^2 is realized in a parametrization-independent way by the lowest (isospinor) $SU(2)_A$ harmonics

此处 $S^2 \sim SU(2)_A/U(1)$，其中 $SU(2)_A$ 是 $\mathcal{N}=2$ 超代数的自同构群。内禀 2 球面 S^2 通过最低阶 (同旋量) $SU(2)_A$ 调和实现，不依赖于参数化选取

$$S^2 \in \left(u_i^+, u_k^-\right),\ u^{+i}u_i^- = 1,\ u_i^\pm \to \mathrm{e}^{\pm i\lambda}u_i^\pm. \tag{25}$$

It is assumed that nothing depends on the $U(1)$ phase $\mathrm{e}^{i\lambda}$, so one effectively deals with the 2-sphere $S^2 \sim SU(2)_A/U(1)$. The superfields given on (24) (harmonic $\mathcal{N}=2$ superfields) are assumed to be expandable in harmonic series on S^2, with the set of all symmetrized products of u_i^+, u_i^- as a basis. These series are fully specified by the $U(1)$ charge of the given superfield.

默认物理量不依赖于 $U(1)$ 相位 $\mathrm{e}^{i\lambda}$，因此我们实际处理的是 2 球面 $S^2 \sim SU(2)_A/U(1)$。定义在 (24) 上的超场 (调和 $\mathcal{N}=2$ 超场) 可以按调和级数在 S^2 上展开，以 u_i^+, u_i^- 的所有对称积为一组基。这类级数完全由给定超场的 $U(1)$ 荷确定。

The main advantage of HSS is the existence of an invariant subspace in it, the $\mathcal{N}=2$ analytic HSS with half of the original odd coordinates

HSS 的主要优势是其中存在一个不变子空间，即保留原奇数坐标一半的 $\mathcal{N}=2$ 解析 HSS

$$\mathbb{HA}^{(4+2|4)} = \left(x_A^m, \theta_A^{+\alpha}, \overline{\theta}_A^{+\dot\alpha}, u^{\pm i}\right) \equiv \left(\zeta, u_i^\pm\right), \tag{26}$$

$$x_A^m = x^m - 2i\theta^{(i}\sigma^m\overline{\theta}^{k)}u_i^+u_k^-,\ \theta^{+\alpha} = \theta^{\alpha i}u_i^+, \overline{\theta}^{+\dot\alpha} = \overline{\theta}^{\dot\alpha i}u_i^+. \tag{27}$$

This is just the $SU(2)$ covariantization of the $O(2)$ analytic superspace (23). It is closed under $\mathcal{N}=2$ supersymmetry transformations and is real with respect to the special involution, which is the product of the ordinary complex conjugation and the antipodal map (Weyl reflection) of S^2. As will be shown later, all $\mathcal{N}=2$ supersymmetric theories admit off-shell formulations in terms of unconstrained superfields defined on (26), the Grassmann analytic $\mathcal{N}=2$ superfields. But before passing to this issue, let us briefly describe the basic technical tools of the harmonic superspace approach.

这正是 $O(2)$ 解析超空间 (23) 的 $SU(2)$ 协变。它在 $\mathcal{N}=2$ 超对称变换下封闭，且对特殊对合是实的，该对合是普通复共轭与 S^2 对径映射 (外尔反射) 的乘积。后文将说明，所有 $\mathcal{N}=2$ 超对称理论都可以离壳表述，采用定义在 (26) 上的无约束超场——格拉斯曼解析 $\mathcal{N}=2$ 超场。但在讨论这个问题之前，我们先简要介绍调和超空间方法的基本技术工具。

Harmonic Calculus on S^2

S^2 上的调和微积分

In the HSS approach, the harmonic sphere $S^2 \sim SU(2)/U(1)$ is coordinatized by "zweibeins" $u^{+i}, u_i^- = \overline{u^{+i}}$ having $SU(2)$ indices i and $U(1)$ charges $\pm$. The solution of the constraint

在 HSS 方法中，调和球面 $S^2 \sim SU(2)/U(1)$ 由具有 $SU(2)$ 指标 i 和 $U(1)$ 电荷 $\pm$ 的“双标架” $u^{+i}, u_i^- = \overline{u^{+i}}$ 坐标化。约束的解

$$u^{+i}u_i^- = 1 \tag{28}$$

is the matrix

是矩阵

$$\| u \| = \begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix} = \frac{1}{\sqrt{1+t\bar{t}}} \begin{pmatrix} e^{i\psi} & -\bar{t}e^{-i\psi} \\ te^{i\psi} & e^{-i\psi} \end{pmatrix},\ 0 \le \psi < 2\pi \tag{29}$$

corresponding to the group $SU(2)$ in the stereographic parametrization $(t, \bar{t}, \psi)$ (one can equally choose any other specific parametrization). To realize the coset space $S^2 \sim SU(2)/U(1)$, the zweibeins have to be defined up to a $U(1)$ phase corresponding to a transformation of the $U(1)$ group in the coset denominator:

对应极射投影参数化 $(t, \bar{t}, \psi)$ 下的群 $SU(2)$ (也可以选择任意其他特定参数化)。为了实现陪集空间 $S^2 \sim SU(2)/U(1)$ ，双标架的定义需相差一个对应陪集分母中 $U(1)$ 群变换的 $U(1)$ 相位:

$$u_i^{+'} = e^{i\alpha} u_i^+,\ u_i^{-'} = e^{-i\alpha} u_i^-. \tag{30}$$

So, the phase ψ in the parametrization (29) is inessential and one effectively deals only with the complex coordinates $t, \bar{t}$. For the phase not to show up at all, the ”functions” on S^2 must possess a definite $U(1)$ charge q and, as a consequence, all the terms in their harmonic expansion must contain only products of zweibeins u^+, u^- of the given charge q . For instance, for $q = +1$

因此，参数化 (29) 中的相位 ψ 是非本质的，实际上我们仅需要处理复坐标 $t, \bar{t}$ 。为了让相位完全不出现，S^2 上的“函数”必须具有确定的 $U(1)$ 电荷 q ，因此其调和展开中的所有项只能包含给定电荷 q 的双标架 u^+, u^- 乘积。例如，对于 $q = +1$

$$f^+(u) = f^i u_i^+ + f^{(ijk)} u_{(i}^+ u_j^+ u_{k)}^- + \dots . \tag{31}$$

Such quantities undergo homogeneous $U(1)$ phase transformations, according to their overall charge. This restriction on the harmonic functions is called $U(1)$ charge preservation. In each term in (31) a complete symmetrization of the indices $i, j, k, \dots$ is assumed. Indeed, any product of the harmonics with a fixed overall $U(1)$ charge can be reduced to their symmetrized product plus lower-rank symmetrized products, using the completeness relation

这类量根据其总电荷经历齐次 $U(1)$ 相位变换。对调和函数的这一限制称为 $U(1)$ 电荷守恒。(31) 的每一项都默认对指标 $i, j, k, \dots$ 做完全对称化。实际上，利用完备性关系，任何固定总 $U(1)$ 电荷的调和乘积都可以约化为对称化乘积加上更低秩的对称化乘积

$$u_i^+ u_k^- - u_k^+ u_i^- = \varepsilon_{ik} \tag{32}$$

following from the basic constraint (28).

这可由基本约束 (28) 推出。

In fact, the zweibeins u_i^+, u_i^- are the fundamental spin 1/2 spherical harmonics, and (31) is an example of a harmonic decomposition on S^2 . This is why u_i^+, u_i^- are referred to as harmonic variables (or simply ”harmonics”).

事实上，双标架 u_i^+, u_i^- 是基本自旋 1/2 球调和函数，而 (31) 是 S^2 上调和分解的一个例子。这就是为什么 u_i^+, u_i^- 被称为调和变量 (简称“调和”)。

It is instructive to list the following specific features of the treatment of the S^2 expansion in the harmonic space approach as compared to the standard textbooks and reviews on harmonic analysis (e.g., in [33,34]).

与调和分析的标准教科书和综述 (例如文献 [33,34]) 相比，列举调和空间方法处理 S^2 展开的以下特点会很有启发。

- The harmonics themselves are regarded as the S^2 coordinates. This allows one to avoid using any explicit parametrization like the stereographic one (29). What really matters is the defining constraint (28) together with the requirement of $U(1)$ charge preservation. If one exploits the harmonics u_i^+, u_i^- as "global" coordinates on S^2 , there arises no need in several "charts" to cover the sphere (which is inevitable when using any explicit parametrization). If one has succeeded in solving some equation in terms of harmonics, then the solution obtained is well defined on the entire sphere.

- 调和本身被视为 S^2 坐标。这使得我们可以避免使用任何显式参数化，比如 (29) 的极射投影参数化。真正重要的是定义性约束 (28)，加上 $U(1)$ 电荷守恒的要求。如果将调和 u_i^+, u_i^- 用作 S^2 上的“整体”坐标，就不需要用多个“坐标卡”覆盖球面 (这在使用任何显式参数化时都是不可避免的)。如果能以调和为变量解出某个方程，得到的解在整个球面上都是良好定义的。

- The harmonic expansions go over symmetrized products of harmonics instead of sets of special functions, like the Jacobi polynomials or the spherical functions. As a result, the coefficients in the harmonic expansions (like $f^i, f^{(ijk)}, \ldots$ in (31)) transform as irreducible representations of the $SU(2)$ group of the coset numerator. This is of special value in $\mathcal{N} = 2$ supersymmetry because the $\mathcal{N} = 2$ supermultiplets are classified according to the $SU(2)$ automorphism group.

- 调和展开是对调和的对称化乘积展开，而非对特殊函数集合 (比如雅可比多项式或球函数) 展开。因此，调和展开的系数 (例如 (31) 中的 $f^i, f^{(ijk)}, \ldots$) 变换为陪集分子 $SU(2)$ 群的不可约表示。这在 $\mathcal{N} = 2$ 超对称中特别有价值，因为 $\mathcal{N} = 2$ 超多重态正是按自同构群 $SU(2)$ 分类的。

In accord with the treatment of $u_i^{\pm}$ as the S^2 coordinates, one may introduce two covariant derivatives compatible with the constraint (28) and having $U(1)$ charges +2 and -2 :

按照将 $u_i^{\pm}$ 视为 S^2 坐标的处理方式，我们可以引入两个满足约束 (28)、且带 $U(1)$ 电荷 +2 和-2 的协变导数:

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} \quad \text{and} \quad D^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}. \tag{33}$$

They play a major role in the harmonic superspace approach and are referred to as harmonic derivatives (This essential use of the derivatives with respect to the additional bosonic coordinates is the characteristic feature of the HSS approach as compared, e.g., with the superfield approach based on the projective $\mathcal{N} = 2$ superspace [35]. As argued in [36], the latter formalism is a particular version of the HSS one corresponding to a special parametrization of the harmonics.). These derivatives commute with the original $SU(2)_A$ group and form, in their own, an $SU(2)$ algebra:

它们在调和超空间方法中发挥核心作用，被称为调和导数 (与例如基于投影 $\mathcal{N}=2$ 超空间的超场方法相比，对额外玻色坐标求导是 HSS 方法的标志性特征。正如文献 [36] 所述，后一种形式体系是 HSS 方法的特殊版本，对应调和的一种特殊参数化)。这些导数与原 $SU(2)_A$ 群对易，其本身构成一个 $SU(2)$ 代数:

$$[D^{++}, D^{--}] = D^0 \equiv u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}, \ \left[D^0, D^{\pm\pm}\right] = \pm 2D^{\pm\pm}, \tag{34}$$

where the operator D^0 also commutes with $SU(2)_A$ and represents the third covariant differential operator on $S^3 \sim SU(2)_A$. When acting on functions on $SU(2)_A/U(1)$ like $f^{(q)}$ in (31), it takes the fixed value,

其中算符 D^0 也与 $SU(2)_A$ 对易，是作用在 $S^3 \sim SU(2)_A$ 上的第三个协变微分算符。当它作用在 (31) 中 $f^{(q)}$ 这类 $SU(2)_A/U(1)$ 上的函数时，会取固定值，

$$D^0 f^{(q)}(u) = q f^{(q)}(u), \tag{35}$$

i.e., it just counts the harmonic $U(1)$ charge. Thus the covariant derivations on the coset S^2 are defined by the operators $D^{\pm\pm}$, in accordance with the dimension of this coset. From the definition (33) follow the obvious rules for the action of $D^{\pm\pm}$ on the harmonics

即它仅用于计数调和 $U(1)$ 电荷。因此，陪集 S^2 上的协变导数由算符 $D^{\pm\pm}$ 定义，这与该陪集的维度一致。由定义 (33) 可直接得到 $D^{\pm\pm}$ 作用在调和上的规则:

$$D^{\pm\pm}u_i^{\pm} = 0, \ D^{\pm\pm}u_i^{\mp} = u_i^{\pm}. \tag{36}$$

To get a better feeling how convenient it is to use $u_i^{\pm}$ as the coordinates of S^2 , let us consider the simple harmonic differential equation

为了更直观地体现将 $u_i^{\pm}$ 用作 S^2 坐标的便捷性，我们来考虑一个简单的调和微分方程

$$D^{++}f^{+}(u) = 0. \tag{37}$$

In harmonics its solution is immediately obtained from (31):

在调和框架下，它的解可由 (31) 直接得到:

$$f^{+} = f^{i}u_i^{+} \tag{38}$$

where f^i are arbitrary constants. Indeed, f^+ has this form because all other terms in its harmonic expansion include u^- . For the general harmonic function $f^{(q)}$ with $q \geq 0$ the solution of the equation analogous to (37) is given by

其中 f^i 是任意常数。事实上，f^+ 取该形式是因为其调和展开中所有其他项都包含 u^- 。对于带 $q \geq 0$ 的一般调和函数 $f^{(q)}$ ，类似 (37) 的方程的解为

$$f^{(q)}(u) = f^{(i_1 i_2 \cdots i_q)} u^+_{i_1} \cdots u^+_{i_q}, \tag{39}$$

where the symmetric tensor $f^{(i_1 i_2 \cdots i_q)}$ represents an irreducible $SU(2)_A$ multiplet with isospin $q/2$. Another important property is

其中对称张量 $f^{(i_1 i_2 \cdots i_q)}$ 对应一个带同位旋 $q/2$ 的不可约 $SU(2)_A$ 多重态。另一重要性质为

$$D^{++} f^{(q)}(u) = 0 \Rightarrow f^{(q)} = 0, \text{ iff } q < 0. \tag{40}$$

It can be easily proved using the harmonic expansions (31) and the property (36).

利用调和展开 (31) 和性质 (36) 可以很容易证明该式。

Finally, in order to be able to construct invariant actions one needs to define an integration on the two-sphere S^2 . In the harmonic approach, it is introduced by the following formal rules:

最后，为了构造不变作用量，我们需要在二维球面 S^2 上定义积分。在调和方法中，积分通过以下形式规则引入:

$$\int du 1 = 1, \int du u_{\left(i_1 .. u^+_{i_k} u^-_{i_{k+1}} .. u^-_{i_{k+l}}\right)} = 0. \tag{41}$$

This definition means the vanishing of the integrals of any spherical function with non-zero isospin (represented by symmetrized products of harmonics). It admits integration by parts, etc. These rules can be justified by the use of some specific parametrization for the harmonics, e.g., (29). In this parametrization, the same harmonic integral is given as

该定义表明，任意带非零同位旋的球函数 (由调和的对称积表示) 的积分均为零，且满足分部积分等性质。这些规则可以通过调和的特定参数化 (例如 (29)) 得到验证。在该参数化下，同一个调和积分可写为

$$\int du f^{(q)}(u) \equiv \frac{i}{4\pi^2} \int_0^{2\pi} d\psi \int \frac{dt \wedge d\bar{t}}{(1 + t\bar{t})^2} f^{(q)}(t, \bar{t}, \psi) . \tag{42}$$

However, the abstract form of the u -integral defined by the rules (41) is most convenient in a field theory.

但在场论中，由规则 (41) 定义的 u 积分的抽象形式是最方便的。

Grassmann Harmonic Analyticity

格拉斯曼调和解析性

The analytic basis in the harmonic $\mathcal{N} = 2$ superspace $\mathbb{H}^{4+2|8}$ is defined as the following set of coordinates

调和 $\mathcal{N} = 2$ 超空间 $\mathbb{H}^{4+2|8}$ 中的解析基定义为如下坐标集合

$$\text{Analytic basis: } \left(x_A^m, \theta^{+\alpha}, \overline{\theta}^{+\dot{\alpha}}, u^{\pm i}, \theta^{-\alpha}, \overline{\theta}^{-\dot{\alpha}}\right) := \left(\zeta, u, \theta^-, \overline{\theta}^-\right), \tag{43}$$

where $x_A^m, \theta^{+\alpha}, \overline{\theta}^{+\dot{\alpha}}$ were defined in (27) and $\theta^{-\alpha} = \theta^{\alpha i} u_i^-, \overline{\theta}^{-\dot{\alpha}} = \overline{\theta}^{\dot{\alpha} i} u_i^-$. The original parametrization (24) is referred to as the ”Central basis”. The main feature of the analytic basis is that it makes manifest the existence of the harmonic analytic subspace (26) closed under the $\mathcal{N} = 2$ supersymmetry transformations. Correspondingly, defining the harmonic projections of the spinor covariant derivatives

其中 $x_A^m, \theta^{+\alpha}, \overline{\theta}^{+\dot{\alpha}}$ 已由式 (27) 定义，$\theta^{-\alpha} = \theta^{\alpha i} u_i^-, \overline{\theta}^{-\dot{\alpha}} = \overline{\theta}^{\dot{\alpha} i} u_i^-$ 同理。原有的参数化方式 (24) 被称为「中心基」。解析基的核心特点是它清晰展现了在 $\mathcal{N} = 2$ 超对称变换下封闭的调和解析子空间 (26) 的存在性。相应地，我们定义旋量协变导数的调和投影

$$D_\alpha^\pm = D_\alpha^i u_i^\pm, \; \bar{D}_{\dot{\alpha}}^\pm = \bar{D}_{\dot{\alpha}}^i u_i^\pm, \tag{44}$$

it is straightforward to find that the derivatives $D_\alpha^+, \bar{D}_{\dot{\alpha}}^+$ become ”short” in the analytic basis

不难发现导数 $D_\alpha^+, \bar{D}_{\dot{\alpha}}^+$ 在解析基中变为「短」形式

$$D_\alpha^+ = \frac{\partial}{\partial \theta^{-\alpha}} := \partial_\alpha^+, \; \bar{D}_{\dot{\alpha}}^+ = \frac{\partial}{\partial \overline{\theta}^{-\dot{\alpha}}} := \partial_{\dot{\alpha}}^+, \tag{45}$$

like, e.g., the $\mathcal{N} = 1$ derivative $\bar{D}_{\dot{\alpha}}$ in the left-chiral basis (cf. (8),(12)). Now, consider a superfield on $\mathbb{H}^{4+2|8}, \Phi^{(q)}\left(x, \theta, \overline{\theta}, u\right)$, where q is the external harmonic $U(1)$ charge, and impose on it the manifestly supersymmetric conditions

例如，就像左手征基中的 $\mathcal{N} = 1$ 导数 $\bar{D}_{\dot{\alpha}}$ 一样 (参见式 (8)、(12))。现在考虑 $\mathbb{H}^{4+2|8}, \Phi^{(q)}\left(x, \theta, \overline{\theta}, u\right)$ 上的超场，其中 q 是外调和 $U(1)$ 荷，对它施加显超对称条件

$$D_\alpha^+ \Phi^{(q)} = 0, \; \bar{D}_{\dot{\alpha}}^+ \Phi^{(q)} = 0. \tag{46}$$

This set of constraints is self-consistent in view of the obvious integrability conditions

根据明显的可积性条件，这组约束是自洽的

$$\left\{D_\alpha^+, D_\beta^+\right\} = \left\{\bar{D}_{\dot{\alpha}}^+, \bar{D}_{\dot{\beta}}^+\right\} = \left\{D_\alpha^+, \bar{D}_{\dot{\beta}}^+\right\} = 0. \tag{47}$$

Since in the analytic basis the derivatives $D_\alpha^+, \bar{D}_{\dot{\alpha}}^+$ are reduced to the partial ones, the conditions (46) become none other than one more example of the Grassmann Cauchy-Riemann conditions: they mean that in this basis the harmonic superfield $\Phi^{(q)}$ does not depend on half of the Grassmann coordinates, viz. $\theta_\alpha^-, \overline{\theta}_{\dot{\alpha}}^-$:

由于解析基中导数 $D_\alpha^+, \bar{D}_{\dot{\alpha}}^+$ 退化为偏导数，条件 (46) 正是格拉斯曼柯西-黎曼条件的又一个例子: 它意味着在该基下，调和超场 $\Phi^{(q)}$ 不依赖一半的格拉斯曼坐标，即 $\theta_\alpha^-, \overline{\theta}_{\dot{\alpha}}^-$:

$$(46) \Rightarrow \Phi^{(q)}\left(x, \theta, \overline{\theta}, u^\pm\right) = \varphi^{(q)}\left(\zeta, u^\pm\right), \tag{48}$$

and so "lives" on the analytic subspace $\mathbb{HA}^{(4+2|4)}$ defined in (26). The superfields $\varphi^{(q)}\left(\zeta^{A},u^{\pm}\right)$ are called "harmonic analytic superfields". This type of Grassmann analyticity clearly generalizes $\mathcal{N}=1$ chirality (cf. (8) and (12)) and is called "Grassmann harmonic analyticity". All $\mathcal{N}=2$ matter, SYM and supergravity theories have adequate off-shell description in terms of the appropriate analytic $\mathcal{N}=2$ superfields.

因此它「生存」在式 (26) 定义的解析子空间 $\mathbb{HA}^{(4+2|4)}$ 上。这样的超场 $\varphi^{(q)}\left(\zeta^{A},u^{\pm}\right)$ 被称为「调和解析超场」。这种格拉斯曼解析性显然是 $\mathcal{N}=1$ 手征性的推广 (参见式 (8) 和 (12)), 被命名为「格拉斯曼调和解析性」。所有 $\mathcal{N}=2$ 物质理论、超杨-米尔斯理论和超引力理论都可以用合适的解析 $\mathcal{N}=2$ 超场给出恰当的离壳描述。

Since the analytic harmonic coordinates $\theta^{+}_{\alpha},\overline{\theta}^{+}_{\dot\alpha}$ carry the harmonic $U(1)$ charge $+1$, the coefficients in the θ expansion of $\varphi^{(q)}\left(x_{A},\theta^{+},\overline{\theta}^{+},u\right)$ have $U(1)$ charges ranging from q to $q-4$, so that the total $U(1)$ charge is always equal to q :

由于解析调和坐标 $\theta^{+}_{\alpha},\overline{\theta}^{+}_{\dot\alpha}$ 携带调和 $U(1)$ 荷 $+1$, $\varphi^{(q)}\left(x_{A},\theta^{+},\overline{\theta}^{+},u\right)$ 的 θ 展开中各项系数的 $U(1)$ 荷取值范围为 q 到 $q-4$ ，因此总 $U(1)$ 荷始终等于 q :

$$\varphi^{(q)}=f^{(q)}(x,u)+\theta^{+\alpha}\psi^{(q-1)}(x,u)+\overline{\theta}^{+}_{\dot\alpha}\chi^{(q-1)\dot\alpha}+\dots. \tag{49}$$

All these component fields are assumed to be expandable in harmonic series of the type (31) on $S^{2}\sim SU(2)_{A}/U(1)$. A very important and surprising property follows from this: the analytic harmonic superfields contain infinitely many fields which can be assembled into infinite series of irreducible supermultiplets with the same fixed superspin Y and increasing superisospins I (with values $I=\left|\frac{q}{2}-1\right|+n$, $n=0,1,2,\dots$) (The superspin is the analog of the Poincaré spin. The superisospin of a given supermultiplet coincides with the isospin of the state with the highest spin (see, e.g.,[14]).). In some cases ($\mathcal{N}=2$ matter) these infinite "tails" of fields become auxiliary while in other cases ($\mathcal{N}=2$ SYM and $\mathcal{N}=2$ SG) they are pure) gauge.

所有这些分量场均可在 $S^{2}\sim SU(2)_{A}/U(1)$ 上按 (31) 型调和级数展开。由此可得一个重要且出人意料的性质: 解析调和超场包含无穷多个场，这些场可组装为一系列不可约超多重态的无穷级数，这些超多重态具有相同的固定超自旋 Y 和递增的超同位旋 I ，取值为 $I=\left|\frac{q}{2}-1\right|+n$ 、 $n=0,1,2,\dots$) (超自旋是庞加莱自旋的类比，给定超多重态的超同位旋与最高自旋态的同位旋一致，参见例如文献 [14])。在某些情况 ($\mathcal{N}=2$ 物质场) 下，这些无穷的场“尾巴”成为辅助场，在另一些情况下则为 ($\mathcal{N}=2$ SYM and $\mathcal{N}=2$ SG) they are pure) 规范场。

The harmonic derivative D^{++} commutes with the spinor derivatives $D^{+}_{\alpha},\bar{D}^{+}_{\dot\alpha}$,

调和导数 D^{++} 与旋量导数 $D^{+}_{\alpha},\bar{D}^{+}_{\dot\alpha}$ 对易,

$$\left[D^{++},D^{+}_{\alpha}\right]=\left[D^{++},\bar{D}^{+}_{\dot\alpha}\right]=0, \tag{50}$$

and so it preserves harmonic analyticity: acting on $\varphi^{(q)}$, it again yields an analytic superfield. In the analytic basis it takes the form

因此它保持调和解析性: 作用于 $\varphi^{(q)}$ 后，所得结果仍是解析超场。它在解析基中取如下形式

$$D_A^{++} = \partial^{++} - 2i\left(\theta^+\sigma^m\bar{\theta}^+\right)\partial_m + \theta^{+\alpha}\partial^+_\alpha + \bar{\theta}^{+\dot{\alpha}}\bar{\partial}^+_{\dot{\alpha}},$$

$$\partial^{++} := u^{+i}\frac{\partial}{\partial u^{-i}} \tag{51}$$

The $U(1)$ -charge counting operator D^0 obviously preserves harmonic analyticity too, in the analytic basis it reads

$U(1)$ 荷计数算符 D^0 显然也保持调和解析性，它在解析基中的形式为

$$D_A^0 = \partial^0 + \theta^{+\alpha}\partial^-_\alpha + \bar{\theta}^{+\dot{\alpha}}\bar{\partial}^-_{\dot{\alpha}} - \theta^{-\alpha}\partial^+_\alpha - \bar{\theta}^{-\dot{\alpha}}\bar{\partial}^+_{\dot{\alpha}},$$

$$\partial^0 := u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}. \tag{52}$$

$\mathcal{N} = 2$ Matter and Gauge Theories in Harmonic Superspace

$\mathcal{N} =$ 调和超空间中的 2 维物质与规范理论

Now we will overview the formulations, main ideas and results related to $\mathcal{N} = 2$ matter and SYM theories in the harmonic superspace approach as a prerequisite to the analogous formulations of $\mathcal{N} = 2$ supergravities.

现在我们将概述调和超空间方法中与 $\mathcal{N} = 2$ 物质和 SYM 理论相关的表述、核心思想与结果，这是推导 $\mathcal{N} = 2$ 超引力类似表述的预备知识。

$\mathcal{N} = 2$ Matter Hypermultiplets

$\mathcal{N} = 2$ 物质超多重态

One of the main results obtained within the HSS approach is the discovery of an off-shell formulation of the Fayet-Sohnius $\mathcal{N} = 2$ hypermultiplet, thus circumventing the no-go theorems which do not seem to allow such a formulation [29,30]. After the HSS formulation has been found, it became clear that the loophole of these theorems was the implicit assumption about the finite number of admissible auxiliary fields.

HSS 方法得到的主要成果之一，是发现了法耶特-索尼乌斯 $\mathcal{N} = 2$ 超多重态的离壳表述，从而规避了不允许这类表述的禁限定理 [29,30]。在找到 HSS 表述后人们才明白，这些定理的漏洞在于对可容许辅助场数量有限这一点的隐含假设。

The basic feature of the off-shell HSS description of the hypermultiplet is the infinite set of auxiliary fields.

超多重态离壳 HSS 描述的基本特征，就是包含无穷多组辅助场。

With the help of the harmonics $u_i^{\pm}$, the constraints (20) can be given another, more suggestive form. Namely, introducing a general $\mathbb{H}^{4+2|8}$ superfield q^+

借助调和 $u_i^{\pm}$ ，约束条件 (20) 可以改写为另一种更直观的形式。也就是说，引入一般的 $\mathbb{H}^{4+2|8}$ 超场 q^+

$$q^+\left(x,\theta,\bar{\theta},u\right)=q^i\left(x,\theta,\bar{\theta}\right)u_i^+ + q^{(ikl)}\left(x,\theta,\bar{\theta}\right)u^+_{(i}u^+_k u^-_{l)}+\ldots, \tag{53}$$

one can equivalently rewrite (20) as

我们可以将 (20) 等价改写为

$$\text{(a) } D^+_\alpha q^+ = \bar{D}^+_{\dot\alpha} q^+ = 0,\ \text{(b) } D^{++}q^+ = 0\,, \tag{54}$$

where $D^+_\alpha = D^i_\alpha u_i^+, \bar{D}^+_{\dot\alpha} = \bar{D}^i_{\dot\alpha} u_i^+$ (recall (44)). Indeed,(54b) implies that $q^+ = q^i u_i^+$, in the same way as (37) implies (38), then (54a) gives just (20):

其中 $D^+_\alpha = D^i_\alpha u_i^+, \bar{D}^+_{\dot\alpha} = \bar{D}^i_{\dot\alpha} u_i^+$ (回顾 (44))。实际上，(54b) 给出 $q^+ = q^i u_i^+$ ，就如同 (37) 导出 (38) 一样，随后 (54a) 恰好给出 (20):

$$D^i q^k u_i^+ u_k^+ = \bar{D}^i_{\dot\alpha} q^k u_i^+ u_k^+ = 0 \;\Rightarrow\; D^{(i}q^{k)} = \bar{D}^{(i}_{\dot\alpha} q^{k)} = 0 \tag{55}$$

(one can take off the symmetric product $u_i^+ u_k^+$ in view of the arbitrariness of the harmonics). The HSS constraints (54) are self-consistent, since the differential operators in them satisfy the integrability conditions (47), (50).

(由于调和的任意性，可以去掉对称乘积 $u_i^+ u_k^+$)。HSS 约束 (54) 是自洽的，因为其中的微分算子满足可积性条件 (47)、(50)。

The advantage of rewriting (20) in the form (54) is revealed in the analytic basis. As explained in section "Grassmann Harmonic Analyticity", in this basis Eqs. (54a) are Grassmann analytic Cauchy-Riemann conditions stating that q^+ is the analytic harmonic $\mathcal{N}=2$ superfield

将 (20) 改写为 (54) 的优势在解析基中体现出来。正如“格拉斯曼调和解析性”一节中所解释的，在该基下方程 (54a) 是格拉斯曼解析柯西-黎曼条件，表明 q^+ 是解析调和 $\mathcal{N}=2$ 超场

$$\text{(54a)} \Rightarrow q^+ = q^+\left(\zeta, u^{\pm}\right). \tag{56}$$

These are purely kinematical conditions, like the $\mathcal{N}=1$ chirality condition. All the dynamical aspects of (20) now prove to be concentrated in (54b)

这些都是纯运动学条件，和 $\mathcal{N}=1$ 手征条件类似。(20) 的所有动力学性质现在都集中在 (54b) 中

$$D_A^{++} q^+(\zeta,u) = \left(\partial^{++} - 2i\theta^+\sigma^m\bar{\theta}^+\partial_m\right) q^+(\zeta,u) = 0 \tag{57}$$

(recall the analytic basis form (51) of D^{++}). It can be easily checked that this equation eliminates all the infinite sets of fields present in the S^2 harmonic expansions of the component fields and gives rise just to the equations of motion (21) for the physical components. The most striking point is that (57) can be derived from the invariant off-shell action

(回顾 D^{++} 的解析基形式 (51))。不难验证，该方程消去了分量场调和展开 S^2 中存在的所有无穷多组场，恰好给出物理分量的运动方程 (21)。最引人注目的一点是，(57) 可以从不变离壳作用量导出

$$S_q^{\text{free}} = -\int du d\zeta^{(-4)} \tilde{q}^+ D^{++} q^+. \tag{58}$$

Here, the operation is a special involution preserving the analytic harmonic super-space (26) (it is reduced to ordinary complex conjugation for the u -independent quantities), and the integration measure of the analytic superspace is defined as

此处，该操作是保持解析调和超空间 (26) 的特殊对合 (对于不依赖 u 的量，它退化为普通复共轭)，解析超空间的积分测度定义为

$$d\zeta^{(-4)} = d^4 x d^2 \theta^+ d^2 \overline{\theta}^+. \tag{59}$$

This measure carries negative $U(1)$ charge because Grassmann integration is equivalent to differentiation with respect to the odd coordinates of the analytic superspace $\theta^+_\alpha, \overline{\theta}^+_{\dot\alpha}$.

该测度带有负的 $U(1)$ 荷，因为格拉斯曼积分等价于对解析超空间 $\theta^+_\alpha, \overline{\theta}^+_{\dot\alpha}$ 的奇坐标求微分。

It is worth pointing out once more that the analytic superfield q^+ is unconstrained in the off-shell action (58), and its harmonic expansion contains an infinite number of auxiliary fields. This is how HSS manages to circumvent the no-go theorem [29, 30] .

值得再次指出，解析超场 q^+ 在离壳作用量 (58) 中不受约束，其调和展开包含无穷多辅助场。HSS 就是通过这一点规避了禁限定理 [29, 30] 。

Now it is rather straightforward to generalize (58) by including general self-interactions. One introduces n hypermultiplet superfields $q^+_a(\zeta, u)$ $\left(\widetilde{(q^+_a)} = \Omega^{ab} q^+_b, \Omega^{ab} = -\Omega^{ba}; a, b = 1, \ldots 2n\right)$ and writes the following general off-shell action:

现在我们可以很直接地推广 (58)，引入一般自相互作用: 引入 n 超多重态超场 $q^+_a(\zeta, u)$ $\left(\widetilde{(q^+_a)} = \Omega^{ab} q^+_b, \Omega^{ab} = -\Omega^{ba}; a, b = 1, \ldots 2n\right)$ ，并写出如下一般离壳作用量:

$$S_q = \int du d\zeta^{(-4)} \left\{ q^+_a D^{++} q^{+a} + L^{+4}(q^+, u^+, u^-) \right\}. \tag{60}$$

Here the indices a, b are raised and lowered by the $Sp(n)$ skew-symmetric tensors $\Omega^{ab}, \Omega_{ab}, \Omega^{ab}\Omega_{bc} = \delta^a_c$. The interaction Lagrangian L^{+4} is an arbitrary function of its arguments, the only restriction is its harmonic $U(1)$ charge $+4$ needed to balance that of the superspace measure. After eliminating the infinite sets of auxiliary fields by their (now nonlinear) equations of motion, one gets the most general self-interaction of n

hypermultiplets. In the bosonic sector, it yields a generic sigma model with a $4n$ -dimensional hyper-Kähler (HK) target manifold in accord with the theorem of Alvarez-Gaumé and Freedman about the one-to-one correspondence between $\mathcal{N}=2$ supersymmetric sigma models and HK manifolds [37] (HK manifolds are $4n$ -dimensional Riemannian manifolds, which admit a triplet of covariantly constant complex structures forming the algebra of quaternionic units or, equivalently, such that their holonomy group lies in $Sp(n)$. In the HSS approach, the triplet of complex structures is parametrized by harmonics [38].). In general, the action (60) and the corresponding HK sigma model possess no isometries. The action (60) is the $\mathcal{N}=2$ analog of the general $\mathcal{N}=1$ matter action (13), the object L^{+4} being the HK potential, analog of the Kähler potential $K\left(\Phi,\overline{\Phi}\right)$ of $\mathcal{N}=1$ supersymmetric sigma models. It encodes the complete information about the local properties of a given HK manifold. Taking some specific L^{+4} , one gets the explicit form of the relevant HK metric after eliminating the auxiliary fields from (60). So, the general hypermultiplet action (60) provides an efficient universal tool for explicit construction of HK metrics. For instance, the well-known 4-dimensional HK Taub-NUT metric corresponds to the choice $L^{+4}\sim(\tilde{q}^{+}q^{+})^{2}$ [14, 39] . Other HK metrics of this kind were constructed in analogous way in [40, 41] . It is also easy to find, e.g., L^{+4} yielding the well-known general Gibbons-Hawking Ansatz [42] for the 4-dimensional HK metrics with at least one tri-holomorphic isometry (i.e., the one commuting with supersymmetry) [14, 38].

此处指标 a,b 由 $Sp(n)$ 反对称张量 $\Omega^{ab},\Omega_{ab},\Omega^{ab}\Omega_{bc}=\delta^{a}_{c}$ 升降。相互作用拉格朗日量 L^{+4} 是其宗量的任意函数，唯一限制是它需带有 +4 的调和 $U(1)$ 荷，以抵消超空间测度的荷。通过辅助场的 (此时为非线性的) 运动方程消去无穷多组辅助场后，我们就得到 n 超多重子最一般的自相互作用。在玻色子部分，根据 Alvarez-Gaumé 和 Freedman 提出的 $\mathcal{N}=2$ 超对称西格玛模型与 HK 流形一一对应的定理 [37]，该自相互作用给出带 $4n$ 维超凯勒 (HK) 目标流形的一般西格玛模型 (HK 流形是 $4n$ 维黎曼流形，它允许一组协变常复结构三元组，该三元组构成四元数单位代数；等价地说，HK 流形的和乐群属于 $Sp(n)$。在 HSS 方法中，复结构三元组由调和参数化 [38])。一般来说，作用量 (60) 和对应的 HK 西格玛模型不具有等距性。作用量 (60) 是一般 $\mathcal{N}=1$ 物质作用量 (13) 的 $\mathcal{N}=2$ 对应物，其中 L^{+4} 就是超凯勒势，对应 $\mathcal{N}=1$ 超对称西格玛模型的凯勒势 $K\left(\Phi,\overline{\Phi}\right)$。它编码了给定 HK 流形全部局域性质的信息。选定特定的 L^{+4} 后，我们就能从 (60) 中消去辅助场，得到对应 HK 度量的显式形式。因此，一般超多重子作用量 (60) 为显式构造 HK 度量提供了高效通用的工具。例如，著名的 4 维 HK 塔布-纽特 (Taub-NUT) 度量对应选择 $L^{+4}\sim(\tilde{q}^{+}q^{+})^{2}$ [14, 39] 。这类 HK 度量的其他例子也通过相同的方式构造于 [40, 41] 。我们也很容易找到能给出著名吉本斯-霍金 (Gibbons-Hawking) 一般 Ansatz[42] 的 L^{+4} ，该 Ansatz 用于描述至少带一个三全纯等距 (即与超对称对易的等距) 的 4 维 HK 度量 [14, 38]。

Following the general recipe of Ref. [43], potential terms (including possible mass terms) can be introduced for any action having at least one tri-holomorphic isometry by introducing a central charge which is identified with the isometry generator times a mass-like parameter (i.e., via a mechanism à la Scherk-Schwarz [44]).

按照文献 [43] 的一般方法，对于任何至少存在一个三全纯等距的作用量，可以通过引入中心荷来引入势项 (包括可能的质量项)，该中心荷可等同于等距生成元乘以类质量参数 (即通过 Scherk-Schwarz 机制 [44] 实现)。

Specifically, this is achieved by extending the harmonic derivative (51) in (60) or (58) as [14]

具体来说，这可以通过如下方式实现: 将 (60) 或 (58) 中的调和导数 (51) 扩展为 [14]

$$D^{++} \to D^{++} + i\left[(\theta^+)^2 - \left(\overline{\theta}\right)^2\right]\frac{\partial}{\partial x^5}, \tag{61}$$

where $\partial/\partial x^5$ is the central charge, and choosing

其中 $\partial/\partial x^5$ 是中心荷，然后选取

$$\frac{\partial}{\partial x^5} q^{+a} = m\lambda^{+a}(q,u), \tag{62}$$

where $\lambda^{+a}(q,u)$ is the Killing vector of the isometry. The form of the scalar potential is completely determined by the Killing vector and the form of the corresponding HK metric. In particular, in the free case $\left(L^{+4}=0\right)$, the only possible effect of the extension (61) is the appearance of mass terms for the physical fields in q^+.

其中 $\lambda^{+a}(q,u)$ 是该等距的基灵矢量。标量势的形式完全由基灵矢量和对应 HK 度量的形式决定。特别地，在自由情形 $\left(L^{+4}=0\right)$ 下，扩展 (61) 唯一可能的效应就是为 q^+ 中的物理场产生质量项。

$\mathcal{N}=2$ Supersymmetric Yang-Mills Theory

$\mathcal{N}=2$ 超对称杨-米尔斯理论

The HSS formulation of $\mathcal{N}=2$ SYM theory reveals surprising affinities with the ordinary ($\mathcal{N}=0$) Yang-Mills theory.

$\mathcal{N}=2$ SYM 理论的 HSS 表述展现出它与普通 ($\mathcal{N}=0$) 杨-米尔斯理论惊人的相似性。

The constraints (22) defining the $\mathcal{N}=2$ SYM theory in the superspace $\mathbb{R}^{4|8}$ can be rewritten in HSS in the following equivalent way

定义超空间 $\mathbb{R}^{4|8}$ 中 $\mathcal{N}=2$ SYM 理论的约束条件 (22) 可以在 HSS 中改写为如下等价形式:

$$\text{(a)}\ \left\{\mathcal{D}^+_\alpha,\mathcal{D}^+_\beta\right\} = \left\{\overline{\mathcal{D}}^+_{\dot\alpha},\overline{\mathcal{D}}^+_{\dot\beta}\right\} = \left\{\mathcal{D}^+_\alpha,\overline{\mathcal{D}}^+_{\dot\beta}\right\} = 0,$$

$$\text{(b)}\ \left[D^{++},\mathcal{D}^+_\alpha\right] = \left[D^{++},\overline{\mathcal{D}}^+_{\dot\alpha}\right] = 0, \tag{63}$$

where $\mathcal{D}^+_\alpha = D^+_\alpha + igA^+_\alpha, \overline{\mathcal{D}}^+_{\dot\alpha} = \bar{D}^+_{\dot\alpha} + ig\bar{A}^+_{\dot\alpha}$ and $A^+_\alpha, \bar{A}^+_{\dot\alpha}$ are some spinor harmonic superfields with values in the algebra of the gauge group, g being the coupling constant. The equivalence of (63) to (22) can be proven quite analogously to the equivalence of (54) and (20). Equation (63b) imply that $\mathcal{D}^+_\alpha = \left(D^l_\alpha + iA^l_\alpha\right)u^+_l$, $\overline{\mathcal{D}}^+_{\dot\alpha} = \left(\bar{D}^k_{\dot\alpha} + i\bar{A}^k_{\dot\alpha}\right)u^+_k$, then (63a) yield (22) by taking off the harmonics $u^+_l u^+_k$.

其中 $\mathcal{D}^+_\alpha = D^+_\alpha + igA^+_\alpha, \overline{\mathcal{D}}^+_{\dot\alpha} = \bar{D}^+_{\dot\alpha} + ig\bar{A}^+_{\dot\alpha}$ 和 $A^+_\alpha, \bar{A}^+_{\dot\alpha}$ 是取值于规范群代数的自旋子调和超场，g 为耦合常数。(63) 与 (22) 的等价性可以用与证明 (54) 和 (20) 等价性十分类似的方法证明。式 (63b) 可推出 $\mathcal{D}^+_\alpha = \left(D^l_\alpha + iA^l_\alpha\right)u^+_l$ 、$\overline{\mathcal{D}}^+_{\dot\alpha} = \left(\bar{D}^k_{\dot\alpha} + i\bar{A}^k_{\dot\alpha}\right)u^+_k$ ，再消去调和项 $u^+_l u^+_k$ 后，就可由 (63a) 得到 (22)。

The constraints (63a) are immediately recognized as integrability conditions for the existence of the gauge-covariant version of Grassmann harmonic analytic superfields (A similar interpretation of $\mathcal{N}=2$ SYM constraints as integrability conditions was given by A. Rosly [45].). Once again, the advantage of the new representation of the $\mathcal{N}=2$ SYM constraints can be revealed by passing to the analytic basis in the superspace $\mathbb{H}^{4+2|8}$ and to a new gauge frame, in which the covariant derivatives $\mathcal{D}^+_\alpha, \mathcal{D}^+_{\dot\beta}$ become "short," thus explicitly solving (63a):

> 不难看出，约束条件 (63a) 就是格拉斯曼调和解析超场规范协变形式存在的可积性条件 (A. Rosly 早已给出 $\mathcal{N}=2$ SYM 约束的类似可积性条件解释 [45])。进入超空间 $\mathbb{H}^{4+2|8}$ 的解析基并变换到新规范架后，$\mathcal{N}=2$ SYM 约束新表示的优势就能再次体现出来: 在新规范架中，协变导数 $\mathcal{D}^+_\alpha, \mathcal{D}^+_{\dot\beta}$ 变为“短形式”，从而显式求解了 (63a):

$$\mathcal{D}^+_\alpha = \partial^+_\alpha,\ \overline{\mathcal{D}}^+_{\dot\beta} = \partial^+_{\dot\beta}. \tag{64}$$

The existence of these basis and frame follows from the general Frobenius theorem. At the same time, the harmonic derivative D^{++} in the new frame acquires a nontrivial gauge connection V^{++} ,

> 该基与规范架的存在性可由一般的弗罗贝尼乌斯定理得到。同时，新规范架中的调和导数 D^{++} 带有非平凡规范联络 V^{++} ,

$$D^{++} \Rightarrow \mathcal{D}^{++} = D^{++} + igV^{++}. \tag{65}$$

The constraint (63b) in the analytic basis and frame is just the statement that V^{++} is the harmonic analytic $\mathcal{N}=2$ superfield

> 解析基和规范架下的约束 (63b) 正是说明 V^{++} 是调和解析 $\mathcal{N}=2$ 超场。

$$V^{++} = V^{++}(\zeta, u) \tag{66}$$

The harmonic analytic gauge connection V^{++} is the fundamental gauge prepotential of $\mathcal{N}=2$ SYM theory. It undergoes the following gauge transformations

> 调和解析规范联络 V^{++} 是 $\mathcal{N}=2$ SYM 理论的基本规范预势。它遵从如下规范变换:

$$\left(V^{++}\right)' = \frac{1}{ig} e^{i\omega}\left(D^{++} + igV^{++}\right) e^{-i\omega}, \tag{67}$$

where $\omega(\zeta, u)$ is an arbitrary analytic gauge parameter. This transformation law resembles the standard gauge transformation of the $\mathcal{N}=0$ Yang-Mills connection. The evident difference is that the latter covariantizes the ordinary x -derivative, while V^{++} covariantizes one of the two derivatives on the internal harmonic sphere $S^2 \sim SU(2)_A/U(1)$.

> 其中 $\omega(\zeta, u)$ 是任意解析规范参数。该变换规律和标准 $\mathcal{N}=0$ 杨-米尔斯联络的规范变换十分相似。明显区别在于，普通杨-米尔斯联络协变化普通 x 导数，而 V^{++} 协变化内部调和球面 $S^2 \sim SU(2)_A/U(1)$ 两个导数中的一个。

The harmonic connection V^{++} contains infinitely many component fields in its combined θ, u expansion, like the hypermultiplet superfield $q^{+}(\zeta, u)$. The difference from $q^{+}(\zeta, u)$ lies, however, in the fact that almost all of these fields are pure gauge degrees of freedom: they can be gauged away by $\omega(\zeta, u)$ which also contains infinitely many components. The finite reminder of $(8+8)$ components is just the off-shell content of the superspin 0, superisospin $0\mathcal{N}=2$ vector multiplet. More precisely, in the proper Wess-Zumino gauge V^{++} has the following form:

调和联络 V^{++} 在其联合 θ, u 展开中包含无穷多分量场，就像超多重态超场 $q^{+}(\zeta, u)$ 一样。但它和 $q^{+}(\zeta, u)$ 的区别在于，几乎所有这些场都是纯规范自由度: 它们可以通过同样包含无穷多分量的 $\omega(\zeta, u)$ 被规范掉。剩余有限的 $(8+8)$ 分量恰好是超自旋 0、超同位旋 $0\mathcal{N}=2$ 向量多重态的离壳内容。更准确地说，在合适的维斯-祖米诺规范下，V^{++} 具有如下形式:

$$V_{WZ}^{++}=\left(\theta^{+}\right)^{2} w\left(x_{A}\right)+\left(\bar{\theta}^{+}\right)^{2} \bar{w}\left(x_{A}\right)+i \theta^{+} \sigma^{m} \bar{\theta}^{+} V_{m}\left(x_{A}\right)+\left(\bar{\theta}^{+}\right)^{2} \theta^{+\alpha} \psi_{\alpha}^{i}\left(x_{A}\right) u_{i}^{-}$$

$$+\left(\theta^{+}\right)^{2} \bar{\theta}_{\dot{\alpha}}^{+} \bar{\psi}^{\dot{\alpha} i}\left(x_{A}\right) u_{i}^{-}+\left(\theta^{+}\right)^{2}\left(\bar{\theta}^{+}\right)^{2} D^{(i j)}\left(x_{A}\right) u_{i}^{-} u_{j}^{-}. \tag{68}$$

Here, $V_{m}, w, \bar{w}, \psi_{i}^{\alpha}, \bar{\psi}^{\dot{\alpha} i}, D^{(i j)}$ are the gauge field, a complex physical scalar field, a doublet of gaugini and a triplet of auxiliary fields, respectively. All the geometric quantities of $\mathcal{N}=2$ SYM theory (spinor and vector connections, covariant superfield strengths, etc), as well as the invariant action, admit a concise representation in terms of the fundamental geometric object $V^{++}(\zeta, u)$. The details of how to construct the invariant action can be found in the book [14] and in the original HSS papers. In particular, the closed V^{++} form of the $\mathcal{N}=2$ SYM action was proposed for the first time in [46].

此处，$V_{m}, w, \bar{w}, \psi_{i}^{\alpha}, \bar{\psi}^{\dot{\alpha} i}, D^{(i j)}$ 分别是规范场、复物理标量场、戈乌尼诺二重态和辅助场三重态。$\mathcal{N}=2$ SYM 理论的所有几何量 (旋量和向量联络、协变超场强度等) 以及不变作用量，都可以用基本几何对象 $V^{++}(\zeta, u)$ 给出简洁表示。构造不变作用量的细节可以在文献 [14] 和原始 HSS 论文中找到。特别地，$\mathcal{N}=2$ SYM 作用量的闭合 V^{++} 形式首次在文献 [46] 中被提出。

The q^{+} hypermultiplet actions (58) or (60) can be coupled to V^{++} in a way quite analogous to how one couples, e.g., fermions to the ordinary Yang-Mills field.

q^{+} 超多重态作用量 (58) 或 (60) 可以按照与费米子耦合到普通杨-米尔斯场非常类似的方式耦合到 V^{++} 。

One should place the superfield q^{+} into the appropriate representation of the gauge group and assume for it the following transformation law

我们需要将超场 q^{+} 放入规范群的合适表示，并假设它满足如下变换规律

$$\delta q_{r}^{+}\left(\zeta, u^{\pm}\right)=i \omega^{A}\left(\zeta, u^{\pm}\right)\left(T^{A}\right)_{r s} q_{s}^{+}\left(\zeta, u^{\pm}\right), \tag{69}$$

where T^{A} are the generators of the gauge group in the given representation and ω^{A} are the corresponding parameters. As usual, one should covariantize the derivatives entering the action. In our case, this is the harmonic derivative in the q^{+} action:

其中 T^A 是给定表示中规范群的生成元，ω^A 是对应的参数。和通常一样，需要对作用量中的导数做协变化。在我们的情形中，它是 q^+ 作用量中的调和导数:

$$D^{++} \Rightarrow \mathcal{D}^{++} = D^{++} + iV^{++A}T^A\,(\zeta, u)\,. \tag{70}$$

The general variety of theories in which $\mathcal{N} = 2$ Yang-Mills fields interact with hypermultiplets is known to contain a subclass of four-dimensional ultraviolet finite quantum field theories (in particular, $\mathcal{N} = 4$ Yang-Mills theory). They also reveal remarkable properties of duality [47]. Harmonic superspace offers a unique possibility to formulate these hybrid theories in a way that $\mathcal{N} = 2$ supersymmetry stays manifest off shell. For instance, the $\mathcal{N} = 4$ super Yang-Mills theory action in the HSS formulation is a sum of the action of V^{++} and that of q^+ in the adjoint representation, with the minimal coupling to V^{++} . The HSS formulation considerably simplifies many aspects and makes manifest many features of such theories, e.g., the proof of non-renormalization theorems, finding out the full structure of the quantum effective actions, etc.

众所周知，在 $\mathcal{N} = 2$ 杨-米尔斯场与超多重态相互作用的各类理论中，存在一个四维紫外有限量子场论的子类 (特别是 $\mathcal{N} = 4$ 杨-米尔斯理论)。它们还展现出非凡的对偶性质 [47]。调和超空间为表述这些混合理论提供了独特可能，能让 $\mathcal{N} = 2$ 超对称在离壳时依然保持显式。例如，HSS 表述中 $\mathcal{N} = 4$ 超杨-米尔斯理论的作用量，就是伴随表示下 V^{++} 的作用量与 q^+ 的作用量之和，并与 V^{++} 存在最小耦合。HSS 表述大幅简化了这类理论的诸多方面，也让其许多性质变得显明，比如非重整化定理的证明、量子有效作用量完整结构的求解等等。

The harmonic approach is also very appropriate for the description of general non-minimal self-couplings of vector $\mathcal{N} = 2$ supermultiplets. These theories are unique because they are the only $\mathcal{N} = 2$ supersymmetric field-theoretical models that admit a natural chiral structure of interactions. For this reason, they may be useful in the phenomenological context as a possible basis of $\mathcal{N} = 2$ GUT models. Sigma models inherent to these couplings are of interest in their own right. Their target manifolds are of the special Kähler type [48,49] and have been discussed, e.g., in connection with the so-called $c*$ -map [50].

调和方法也非常适合描述向量 $\mathcal{N} = 2$ 超多重态的一般非最小自耦合。这些理论是唯一具备自然手征相互作用结构的 $\mathcal{N} = 2$ 超对称场论模型，因此十分特殊。由于这个原因，它们作为 $\mathcal{N} = 2$ 大统一模型的可能基础，在唯象学背景下是有价值的。这类耦合固有的 sigma 模型本身就很有研究意义。它们的目标流形是特殊凯勒型 [48,49]，已有相关讨论，例如联系所谓的 $c*$ -映射 [50]。

$\mathcal{N} = 2$ Einstein Supergravity ”From Scratch”

$\mathcal{N} = 2$ 爱因斯坦超引力“从零开始”

The first example of off-shell $\mathcal{N} = 2$ Einstein supergravity was derived at the component level in [51-53]. As was shown in [18], in the HSS formulation it corresponds to choosing the so-called nonlinear multiplet as a compensator in conformal $\mathcal{N} = 2$ supergravity. Actually, this kind of $\mathcal{N} = 2$ Einstein SG can be deduced in the HSS approach without referring to conformal SG [15], directly from the preservation of the harmonic Grassmann analyticity and the appropriate choice of the fundamental gauge group, much like to $\mathcal{N} = 2$ SYM theory (An attempt in a similar direction was also undertaken in [54].). Here we give a brief derivation of this theory following Ref. [15].

离壳 $\mathcal{N} = 2$ 爱因斯坦超引力的第一个实例是在文献 [51-53] 中于分量层级推导出来的。正如文献 [18] 所示，在调和超空间 (HSS) 表述中，它对应于在共形 $\mathcal{N} = 2$ 超引力中选择所谓的非线性多重态作为补偿子。实际上，这类 $\mathcal{N} = 2$ 爱因斯坦超引力 (SG) 可以在调和超空间方法中不借助共形超引力，直接从调和格拉斯曼解析性的保持和对基本规范群的恰当选择推导出来，这与 $\mathcal{N} = 2$ 超对称杨-米尔斯 (SYM) 理论非常相似 (文献 [54] 也做过类似方向的尝试。)。本文我们将参照文献 [15] 给出该理论的一个简要推导。

From Central to Analytic Bases

从中心基到解析基

The starting point of the consideration in [15] is the so called τ gauge group acting on the $\mathcal{N} = 2$ superspace coordinates,

文献 [15] 研究的出发点是作用在 $\mathcal{N} = 2$ 超空间坐标上的所谓 τ 规范群,

$$Z^M = \left(x^m, x^5, \theta^{\mu i}, \overline{\theta}^{\dot{\mu}k}\right),$$

$$\delta x^m = \tau^m(Z), \ \delta x^5 = \tau^5(Z), \ \delta\theta^{\hat{\mu}i} = \tau^{\hat{\mu}i}(Z), \ \hat{\mu} = (\mu, \dot{\mu}). \tag{71}$$

The constraints of $\mathcal{N} = 2$ Einstein SG can be then written as

$\mathcal{N} = 2$ 爱因斯坦超引力的约束可以写为

$$\left\{\mathcal{D}^i_\alpha, \mathcal{D}^k_\beta\right\} = \varepsilon_{\alpha\beta}\varepsilon^{ik}\mathcal{D}_5 + \text{ curvature (and c.c.)}, \tag{72}$$

$$\left\{\mathcal{D}^i_\alpha, \overline{\mathcal{D}}_{k\dot{\beta}}\right\} = \delta^i_k\mathcal{D}_{\alpha\dot{\beta}} + \text{ curvature }, \tag{73}$$

where the indices $\alpha, \dot{\beta}, i$ of the spinor covariant derivatives are transformed by the local tangent space Lorentz group $SL(2, C)$ with parameters $\Omega^{(\alpha\beta)}(Z), \overline{\Omega}^{(\dot{\alpha}\dot{\beta})}(Z)$ and global automorphism group $SU(2)_{\text{aut}}$ with parameters $\tau^{(ik)}$. The bosonic covariant derivative $\mathcal{D}_{\alpha\dot{\beta}}$ is the proper covariantization of the derivative ∂_m. The gauge group parameters are assumed to be independent of x^5, so $\mathcal{D}_5$ is constrained to reduce to the partial derivative

其中旋量协变导数的 $\alpha, \dot{\beta}, i$ 指标由参数为 $\Omega^{(\alpha\beta)}(Z), \overline{\Omega}^{(\dot{\alpha}\dot{\beta})}(Z)$ 的局部切空间洛伦兹群 $SL(2, C)$ 和参数为 $\tau^{(ik)}$ 的整体自同构群 $SU(2)_{\text{aut}}$ 变换。玻色协变导数 $\mathcal{D}_{\alpha\dot{\beta}}$ 是导数 ∂_m 的恰当协变化。规范群参数假定与 x^5 无关，因此 $\mathcal{D}_5$ 受约束退化为偏导数

$$\mathcal{D}_5 = \frac{\partial}{\partial x^5}. \tag{74}$$

In fact, the actual constraints are only those parts of the relations (72), (73), which are symmetric in the $SU(2)$ indices, i.e.,

实际上，真正的约束只是式 (72)、(73) 中关于 $SU(2)$ 指标对称的那些部分，即

$$\left\{\mathcal{D}_{\hat{\alpha}}^{(i)}, \mathcal{D}_{\hat{\beta}}^{k\}}\right\} = 0,\ \hat{\alpha} := (\alpha, \dot{\alpha}). \tag{75}$$

The traces just yield the definition of $\mathcal{D}_{\alpha\dot{\beta}}, \mathcal{D}_5$.

迹部分仅给出 $\mathcal{D}_{\alpha\dot{\beta}}, \mathcal{D}_5$ 的定义。

At the next step, harmonics come into play

下一步，就该引入调和变量了

$$Z^M \to \left\{Z^M, u^{\pm i}\right\} = \left\{\left(x^m, x^5, \theta^{\hat{\mu}i}\right), u^{\pm i}\right\}. \tag{76}$$

One defines the harmonic projections of the spinor derivatives $\mathcal{D}^i_{\hat{\alpha}} \to \mathcal{D}^{\pm}_{\hat{\alpha}}, \mathcal{D}^{\pm}_{\hat{\alpha}} = u^{\pm}_i \mathcal{D}^i_{\hat{\alpha}}$ and write the essential part of (72),(73) as

我们定义旋量导数 $\mathcal{D}^i_{\hat{\alpha}} \to \mathcal{D}^{\pm}_{\hat{\alpha}}, \mathcal{D}^{\pm}_{\hat{\alpha}} = u^{\pm}_i \mathcal{D}^i_{\hat{\alpha}}$ 的调和投影，并将式 (72)、(73) 的核心部分写为

$$\left\{\mathcal{D}^+_{\hat{\alpha}}, \mathcal{D}^+_{\hat{\beta}}\right\} = \text{ curvature.} \tag{77}$$

Clearly, this is just the integrability condition for the existence of the covariantly analytic harmonic $\mathcal{N} = 2$ superfields

显然，这正是协变解析调和 $\mathcal{N} = 2$ 超场存在的可积性条件

$$\mathcal{D}^+_{\hat{\alpha}} \Phi(Z, u) = 0 \tag{78}$$

Like in the case of $\mathcal{N} = 2$ SYM theory, this implies the existence of the basis where this analyticity gets manifest. To accomplish this, we need to extend (77) to the so called **CR** ('Cauchy-Riemann") structure [55], from which both the linearity of $\mathcal{D}^{\pm}_{\hat{\alpha}}$ in the harmonic variables and the original constraints in the "central basis" follow. Such a structure is obtained by adding, to the relation (77), the following set of the relations involving the harmonic derivatives

和 $\mathcal{N} = 2$ 超对称杨-米尔斯理论的情况一样，这说明存在一个能让该解析性显式呈现的基。为了实现这一点，我们需要将式 (77) 推广为所谓的 **CR** (“柯西-黎曼”) 结构 [55]，由此既可以得到 $\mathcal{D}^{\pm}_{\hat{\alpha}}$ 在调和变量中的线性性，也可以得到“中心基”下的原始约束。该结构通过在式 (77) 中加入以下包含调和导数的关系得到:

$$\text{(a) } \left[\mathcal{D}^{++}, \mathcal{D}^+_{\hat{\alpha}}\right] = 0,\ \text{(b) } \left[\mathcal{D}^0, \mathcal{D}^+_{\hat{\alpha}}\right] = \mathcal{D}^+_{\hat{\alpha}}\ ,$$

$$\text{(c) } \left[\mathcal{D}^0, \mathcal{D}^{++}\right] = 2\mathcal{D}^{++}\ . \tag{79}$$

In the central basis, where $\mathcal{D}^{++} = D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}$ and $\mathcal{D}^{0} = D^{0} = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}$, the constraints (79a, b) indeed imply $\mathcal{D}^{+}_{\hat{\alpha}} = u^{+}_{i}\mathcal{D}^{i}_{\hat{\alpha}}$ and, after substitution this in (77) and taking off the harmonics, the original constraints (75) are recovered.

在中心基中，当满足条件 $\mathcal{D}^{++} = D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}$ 和 $\mathcal{D}^{0} = D^{0} = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}$ 时，约束 (79a, b) 确实给出 $\mathcal{D}^{+}_{\hat{\alpha}} = u^{+}_{i}\mathcal{D}^{i}_{\hat{\alpha}}$ ，将其代入式 (77) 并消去调和变量后，原始约束 (75) 就得到了恢复。

The main merit of singling out the above **CR** structure is the opportunity to pass to the new, analytic basis, in which just the spinor derivatives $\mathcal{D}^{+}_{\hat{\alpha}}$ become short. This is achieved through introducing the proper "bridges" to the analytic superspace coordinates, that allows one to reach the "almost simple" form for $\mathcal{D}^{+}_{\hat{\alpha}}$

分离出上述 **CR** 结构的主要优点在于，我们可以由此过渡到新的解析基，在该基中旋量导数 $\mathcal{D}^{+}_{\hat{\alpha}}$ 会变得简洁。这通过引入恰当的“桥”连接到解析超空间坐标实现，它能让 $\mathcal{D}^{+}_{\hat{\alpha}}$ 变为“近简”形式

$$\mathcal{D}^{+}_{\hat{\alpha}} = E^{\hat{\mu}}_{\hat{\alpha}}\frac{\partial}{\partial\theta^{-\hat{\mu}}_{A}} + A^{+}_{\hat{\alpha}} \equiv \nabla^{+}_{\hat{\alpha}} + A^{+}_{\hat{\alpha}}, \tag{80}$$

where $E^{\hat{\mu}}_{\hat{\alpha}}$ are the only remaining vielbeins. The analytic basis of the harmonic superspace in the curved case is constituted by the coordinates

其中 $E^{\hat{\mu}}_{\hat{\alpha}}$ 是仅存的标架。弯曲情况下调和超空间的解析基由以下坐标构成

$$\{Z_A, u^{\pm}_i\} = \left\{\left(x^m_A, x^5_A, \theta^{+\hat{\mu}}_A, u^{\pm}_i\right), \theta^{-\hat{\mu}}_A\right\} \equiv \left(\zeta, x^5_A, u^{\pm}_i, \theta^{-\hat{\mu}}_A\right), \tag{81}$$

such that the curved analytic subspace, $\left(\zeta, x^5_A, u^{\pm}_i\right) := \left(x^m_A, \theta^{+\hat{\mu}}_A, x^5_A, u^{\pm}_i\right)$, is preserved by the action of the analytic basis gauge group

使得弯曲解析子空间 $\left(\zeta, x^5_A, u^{\pm}_i\right) := \left(x^m_A, \theta^{+\hat{\mu}}_A, x^5_A, u^{\pm}_i\right)$ 在解析基规范群的作用下保持不变

$$\delta x^{m,5}_A = \lambda^{m,5}(z_A, u),\ \delta\theta^{+\hat{\mu}}_A = \lambda^{+\hat{\mu}}(z_A, u),\ \delta\theta^{-\hat{\mu}}_A = \lambda^{\hat{\mu}-}(z_A, u, \theta^{-}_A), \tag{82}$$

$$\delta u^{\pm}_i = 0. \tag{83}$$

The indices $\hat{\alpha}$ of (80) are still rotated by the local Lorentz parameters $\Omega^{(\alpha\beta)}(Z), \overline{\Omega}^{(\dot{\alpha}\dot{\beta})}(Z)$, which are covariantly $u^{\pm}_i$ independent,

式 (80) 中 $\hat{\alpha}$ 指标仍由局域洛伦兹参数 $\Omega^{(\alpha\beta)}(Z), \overline{\Omega}^{(\dot{\alpha}\dot{\beta})}(Z)$ 转动，这些参数满足协变 $u^{\pm}_i$ 无关性

$$\mathcal{D}^{++}\Omega^{(\alpha\beta)} = \mathcal{D}^{++}\overline{\Omega}^{(\dot{\alpha}\dot{\beta})} = 0.$$

So the vielbein $E_{\hat{\alpha}}{}^{\hat{\mu}}$ and the gauge connection $A^{+}_{\hat{\alpha}}$ in (80) have the following transformation laws

因此式 (80) 中的标架 $E_{\hat{\alpha}}{}^{\hat{\mu}}$ 与规范联络 $A^{+}_{\hat{\alpha}}$ 满足如下变换规律

$$\delta E_{\hat{\alpha}}^{\hat{\mu}} = \Omega_{\hat{\alpha}}^{\hat{\beta}} E_{\hat{\beta}}^{\hat{\mu}} + E_{\hat{\alpha}}^{\hat{\upsilon}} \partial_{\hat{\upsilon}}^{+} \lambda^{-\hat{\mu}}, \ \partial_{\hat{\upsilon}}^{\pm} := \frac{\partial}{\partial \theta^{\mp \hat{\mu}}}, \tag{84}$$

$$\delta A_{\hat{\alpha}\hat{\beta}\hat{\gamma}}^{+} = -\nabla_{\hat{\alpha}}^{+} \Omega^{\beta\gamma} + \Omega_{\hat{\alpha}}^{\hat{\rho}} A_{\hat{\rho}\hat{\beta}\hat{\gamma}}^{+} + \Omega_{\hat{\beta}}^{\hat{\rho}} A_{\hat{\alpha}\hat{\rho}\hat{\gamma}}^{+} + \Omega_{\hat{\gamma}}^{\hat{\rho}} A_{\hat{\alpha}\hat{\beta}\hat{\rho}}^{+}. \tag{85}$$

Here,

在此,

$$\Omega_{\hat{\alpha}}^{\hat{\beta}} = \begin{pmatrix} \Omega_{(\alpha}^{\ \beta)} & 0 \\ 0 & \overline{\Omega}_{(\dot{\alpha}}^{\ \dot{\beta})} \end{pmatrix}. \tag{86}$$

In the analytic basis, the spinor derivative $\mathcal{D}_{\hat{\alpha}}^{+}$ becomes almost short, while the harmonic derivative $\mathcal{D}^{++}$ acquires nontrivial vielbeins,

在解析基中，旋量导数 $\mathcal{D}_{\hat{\alpha}}^{+}$ 几乎缩并，而调和导数 $\mathcal{D}^{++}$ 会得到非平凡标架

$$\mathcal{D}_{AB}^{++} = \partial^{++} + H^{++m,5} \partial_{m,5} + H^{++\hat{\mu}\pm} \partial_{\hat{\mu}}^{\mp} \equiv \partial^{++} + H^{++M} \partial_{M}, \tag{87}$$

$$\delta H^{++M} = \mathcal{D}^{++} \lambda^{M}, \ M = (m, 5, \hat{\mu}\pm). \tag{88}$$

The basic constraint (79a) implies the following conditions for the harmonic vielbeins

基本约束 (79a) 对调和标架给出如下条件

$$\nabla_{\hat{\alpha}}^{+} H^{++m,5} \partial_{m,5} + \nabla_{\hat{\alpha}}^{+} H^{++\hat{\mu}+} \partial_{\hat{\mu}}^{-} + \left(\mathcal{D}^{++} E_{\hat{\alpha}}^{\hat{\mu}} - \nabla_{\hat{\alpha}}^{+} H^{++\hat{\mu}-} \right) \partial_{\hat{\mu}}^{+}$$

$$-\mathcal{D}^{++} A_{\hat{\alpha}}^{+} = 0 \tag{89}$$

The obvious consequence is the analyticity of the basic vielbeins

其直接推论就是基本标架的解析性

$$\nabla_{\hat{\alpha}}^{+} H^{++m,5} = \nabla_{\hat{\alpha}}^{+} H^{++\hat{\mu}+} = 0 \Rightarrow$$

$$H^{++m,5} = H^{++m,5} (\zeta, u), \ H^{++\hat{\mu}+} = H^{++\hat{\mu}+} (\zeta, u). \tag{90}$$

As for $H^{++\hat{\mu}-}$, it can be gauged away into its flat limit, using the fact that the parameter $\lambda^{\hat{\mu}-}$ is a general harmonic superfunction,

至于 $H^{++\hat{\mu}-}$ ，利用参数 $\lambda^{\hat{\mu}-}$ 是一般调和超函数这一性质，我们可以通过规范变换将其消去，得到它的平坦极限

$$H^{++\hat{\mu}-}=\theta_A^{+\hat{\mu}},\ \mathcal{D}^{++}\lambda^{-\hat{\mu}}=\lambda^{+\hat{\mu}}. \tag{91}$$

Now, recalling the gauge transformations (88), we can cast the remaining analytic vielbeins in the following Wess-Zumino form

现在，回顾规范变换 (88)，我们可以将剩余解析标架写为如下韦斯-朱米诺形式

$$H^{++m}(\zeta,u)=-2i\theta^{+}\sigma^{a}\overline{\theta}^{+}e_a^m+\kappa\left(\overline{\theta}^{+}\right)^2\theta^{+\mu}\psi_{\mu i}^{m}u^{-i}$$

$$+\kappa(\theta^{+})^2\overline{\theta}_{\dot{\mu}}^{+}\overline{\psi}_i^{m\dot{\mu}}u^{-i}+\kappa(\theta^{+})^2\left(\overline{\theta}^{+}\right)^2V_{ij}^{m}u^{-i}u^{-j},$$

$$H^{++\mu+}(\zeta,u)=\kappa(\theta^{+})^2\overline{\theta}_{\dot{\mu}}^{+}\left(A^{\mu\dot{\mu}}+iV^{\mu\dot{\mu}}\right)+\kappa\left(\overline{\theta}^{+}\right)^2\theta^{+\upsilon}\left[\delta_{\upsilon}^{\mu}(M+iN)+T_{(\upsilon}^{\mu)}\right]$$

$$+\kappa(\theta^{+})^2\left(\overline{\theta}^{+}\right)^2\xi_i^{\mu}u^{-i},\ H^{++\dot{\mu}+}=\widetilde{H^{++\mu+}}$$

$$H^{++5}(\zeta,u)=i\left[(\theta^{+})^2-\left(\overline{\theta}^{+}\right)^2\right]+i\kappa\theta^{+}\sigma^{a}\overline{\theta}^{+}B_a+\kappa\left(\overline{\theta}^{+}\right)^2\theta^{+\mu}\rho_{\mu}^{i}u_i^{-}$$

$$+\kappa(\theta^{+})^2\overline{\theta}^{+\dot{\mu}}\overline{\rho}_{\dot{\mu}}^{i}+\kappa(\theta^{+})^2\left(\overline{\theta}^{+}\right)^2S^{(ij)}u_i^{-}u_j^{-}, \tag{92}$$

κ being Newton’s constant ($[\kappa]=-1$) . Here the fields $e_a^m,\psi_{\mu i}^m,\overline{\psi}_{\dot{\mu}}^{mi},B_a$ describe the graviton, gravitini, and ”graviphoton” (gauge field for the central charge local shifts), all in the ”frame-like” formulation. All other fields are auxiliary. With taking into account the residual gauge freedom of (92) (local diffeomorphisms, local Lorentz transformations and local supersymmetry) we are left with $(40+40)$ essential degrees of freedom, that is just the off-shell field content of the simplest $\mathcal{N}=2$ SG [51,52].

其中 κ 是牛顿常数 ($[\kappa]=-1$)。这里场 $e_a^m,\psi_{\mu i}^m,\overline{\psi}_{\dot{\mu}}^{mi},B_a$ 描述引力子、引力微子和“引力光子”(中心荷局域平移的规范场)，全部都在类标架表述中。所有其他场都是辅助场。考虑到 (92) 的剩余规范自由度 (局域微分同胚、局域洛伦兹变换和局域超对称性)，我们最终得到 $(40+40)$ 个基本自由度，这正是最简 $\mathcal{N}=2$ 超引力 [51,52] 的脱壳场内容。

Our next point is singling out the remaining constraints from (89). Using (91), the coefficient of $\partial_{\hat{\mu}}^{+}$ yields

接下来我们要提取出 (89) 中剩余的约束。利用 (91)，$\partial_{\hat{\mu}}^{+}$ 的系数给出

$$\mathcal{D}^{++}E_{\hat{\alpha}}^{\hat{\mu}}=0, \tag{93}$$

which means the covariant harmonic independence of $E_{\hat{\alpha}}^{\hat{\mu}}$. Because of this property one can accomplish the further gauge-fixing by making use of the covari-antly u -independent parameters $\Omega_{\alpha\beta}$. The corresponding holomorphic and anti-holomorphic traceless parts of the diagonal in the matrix $E_{\hat{\alpha}}^{\hat{\mu}}$ can be gauged away, after which the vielbein can be reduced to the form

这意味着 $E_{\hat{\alpha}}^{\hat{\mu}}$ 满足协变调和无关性。利用这一性质，我们可以通过协变 u 无关的参数 $\Omega_{\alpha\beta}$ 进一步固定规范。矩阵 $E_{\hat{\alpha}}^{\hat{\mu}}$ 对角元的全纯和反全纯无迹部分都可以通过规范变换消去，之后标架可以约化为如下形式

$$E_{\hat{\alpha}}^{\hat{\mu}} = \begin{pmatrix} F\delta_{\alpha}^{\mu} & FF_{\alpha}^{\dot{\mu}} \\ \widetilde{F}\widetilde{F}_{\dot{\alpha}}^{\mu} & \widetilde{F}\delta_{\dot{\alpha}}^{\dot{\mu}} \end{pmatrix}. \tag{94}$$

Respectively, the Lorentz transformations in this gauge get induced by the coordinate ones

相应地，该规范下的洛伦兹变换由坐标变换诱导得出

$$\Omega_{\alpha\beta} = -\Delta^{+}_{(\alpha}\lambda^{-}_{\beta)}, \ \Delta^{+}_{\alpha} := \frac{1}{F}\nabla^{+}_{\alpha} = \partial^{+}_{\alpha} + F_{\alpha}^{\dot{\mu}}\overline{\partial}^{+}_{\dot{\mu}}. \tag{95}$$

Finally, (89) yields the harmonic independence of the connection

最终，(89) 给出联络满足调和无关性

$$\mathcal{D}^{++}A^{+}_{\hat{\alpha}} = 0 \tag{96}$$

There still remains the constraint (77). The vanishing of the torsion in this anticommutator amounts to the relations

仍然需要考虑约束 (77)。这个对易子中挠率为零等价于如下关系

$$\text{(a) } \mathcal{D}^{+}_{(\alpha}E_{\beta)}^{\hat{\mu}} = 0, \ \text{(b) } \mathcal{D}^{+}_{\alpha}E_{\dot{\beta}}^{\hat{\mu}} + \overline{\mathcal{D}}^{+}_{\dot{\beta}}E_{\alpha}^{\hat{\mu}} = 0 \ \text{(and c.c.)}, \tag{97}$$

which allow one to express $A^{+}_{\hat{\alpha}}$ through the entries of (94)

这些关系允许我们将 $A^{+}_{\hat{\alpha}}$ 用式 (94) 中的各项表示出来

$$\text{(a) } A^{+}_{\alpha\beta\gamma} = 2\varepsilon_{\alpha(\beta}\Delta^{+}_{\gamma)}F \ \text{,(b) } A^{+}_{\alpha\beta\dot{\gamma}} = -2\left(\nabla^{+}_{\alpha}E_{(\beta}^{\hat{\mu}}\right)E^{-1}_{\hat{\mu}\dot{\gamma}} - 2\left(\overline{\nabla}^{+}_{(\dot{\beta}}E_{\alpha}^{\hat{\mu}}\right)E^{-1}_{\hat{\mu}\dot{\gamma})}\,. \tag{98}$$

Indeed, substituting (94) into (97a) immediately yields (98a) and also implies the constraint

事实上，将 (94) 代入 (97a) 即可直接得到 (98a)，同时还给出约束条件

$$\Delta^{+}_{(\alpha}F_{\beta)}^{\mu} = 0 \Rightarrow \left\{\Delta^{+}_{\alpha}, \Delta^{+}_{\beta}\right\} = 0. \tag{99}$$

Note that after fixing the ”soldering” gauge (94), the actual difference between the tangent space and the ”world” spinor indices becomes to some extent elusive, so in what follows we will use the same letters to denote them.

注意，固定好“焊接”规范 (94) 后，切空间与“世界”旋量指标的实际差别在一定程度上变得模糊，因此在下文中我们将使用相同字母来标记二者。

Covariant Derivatives $\mathcal{D}^{--}$ and $\mathcal{D}^{-}_{\hat{\alpha}}$; Further Torsion Constraints

协变导数 $\mathcal{D}^{--}$ 和 $\mathcal{D}^{-}_{\hat{\alpha}}$ ；进一步的挠率约束

To finish the differential geometry routine, it remains to find the proper expressions through analytic potentials for the quantities F and $F_\alpha{}^{\hat{\mu}}$ obeying the constraints (93) and (99). This can be done after defining the second (not preserving the analyticity) harmonic derivative, $\mathcal{D}^{--}$.

为完成微分几何流程，还需通过解析势求出满足约束 (93) 和 (99) 的量 F 和 $F_\alpha{}^{\hat{\mu}}$ 的恰当表达式。这一步需要在定义第二个不保持解析性的调和导数 $\mathcal{D}^{--}$ 之后完成。

The basic relation defining $\mathcal{D}^{--}$ is the direct analog of the harmonic zero curvature condition of $\mathcal{N} = 2$ SYM theory:

定义 $\mathcal{D}^{--}$ 的基本关系是 $\mathcal{N} = 2$ 杨-米尔斯理论调和零曲率条件的直接类比:

$$[\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^0 = \partial^0 + \theta^{+\hat{\mu}}\partial^-_{\hat{\mu}} - \theta^{-\hat{\mu}}\partial^+_{\hat{\mu}}. \tag{100}$$

After introducing the appropriate vielbeins in the analytic basis,

在解析基中引入合适的标架后,

$$\mathcal{D}^{--}_{AB} = \partial^{--} + H^{--M}\partial_M = \partial^{--} + H^{--m,5}\partial_{m,5} + H^{--\hat{\mu}\pm}\partial^{\mp}_{\hat{\mu}},$$

$$\delta H^{--M} = \mathcal{D}^{--}\lambda^M \tag{101}$$

the relation (100) gives rise to the equations

关系式 (100) 导出了如下方程

$$\mathcal{D}^{++}H^{--m,5} - \mathcal{D}^{--}H^{++m,5} = 0,$$

$$\mathcal{D}^{++}H^{--\hat{\mu}\pm} - \mathcal{D}^{--}H^{++\hat{\mu}\pm} = \pm\theta^{\pm\hat{\mu}}_A, \tag{102}$$

which can be solved to express the negatively charged vielbeins in terms of the basic positively charged analytic ones.

求解该方程即可将负电荷标架表示为基本正电荷解析标架的函数。

The next important step is the definition of the second covariant spinorial derivative

下一个重要步骤是定义第二个协变旋量导数

$$\mathcal{D}^-_{\hat{\alpha}} = \left[\mathcal{D}^{--}, \mathcal{D}^+_{\hat{\alpha}}\right], \; \left[\mathcal{D}^{++}, \mathcal{D}^-_{\hat{\alpha}}\right] = \mathcal{D}^+_{\hat{\alpha}}. \tag{103}$$

From Eqs. (80) and (101), it then follows

由式 (80) 和 (101) 可得

$$\mathcal{D}^-_{\hat{\alpha}} = -\nabla^+_{\hat{\alpha}} H^{--\hat{\mu}\pm}\partial^{\mp}_{\hat{\mu}} - \nabla^+_{\hat{\alpha}} H^{--m,5}\partial_{m,5} + A^-_{\hat{\mu}},\ A^-_{\hat{\mu}} = \mathcal{D}^{--}A^+_{\hat{\mu}}, \tag{104}$$

where we used the property that Eq. (93) implies $\mathcal{D}^{--}E^{\hat{\mu}}_{\hat{\alpha}} = 0$ (that can be easily proved in the central basis). The vector connection is defined by the standard relation

其中我们利用了式 (93) 蕴含 $\mathcal{D}^{--}E^{\hat{\mu}}_{\hat{\alpha}} = 0$ 这一性质 (该性质可在中心基中轻易证明)。矢量联络由标准关系式定义为

$$\mathcal{D}_{\alpha\dot{\alpha}} = \frac{1}{2}\left(\left\{\mathcal{D}^+_{\alpha}, \overline{\mathcal{D}}^-_{\dot{\alpha}}\right\} - \left\{\mathcal{D}^-_{\alpha}, \overline{\mathcal{D}}^+_{\dot{\alpha}}\right\}\right). \tag{105}$$

It is convenient to rewrite the anticommutation relations involving $\mathcal{D}^-_{\hat{\alpha}}$ as

将包含 $\mathcal{D}^-_{\hat{\alpha}}$ 的对易关系改写为如下形式会更方便:

(a) $\left\{\mathcal{D}^-_{\alpha}, \mathcal{D}^+_{\beta}\right\} = \varepsilon_{\alpha\beta}\partial_5 +$ curvature (and c.c.); (b) $\left\{\mathcal{D}^{\pm}_{\alpha}, \bar{\mathcal{D}}^{\mp}_{\dot{\beta}}\right\} = \pm\mathcal{D}_{\alpha\dot{\beta}} +$ curvature.

(a) $\left\{\mathcal{D}^-_{\alpha}, \mathcal{D}^+_{\beta}\right\} = \varepsilon_{\alpha\beta}\partial_5 +$ 曲率 (及其复共轭); (b) $\left\{\mathcal{D}^{\pm}_{\alpha}, \bar{\mathcal{D}}^{\mp}_{\dot{\beta}}\right\} = \pm\mathcal{D}_{\alpha\dot{\beta}} +$ 曲率。

(106)

Now we are prepared to express the basic objects $F, F^{\hat{\mu}}_{\alpha}$ through the negatively charged vielbeins $H^{--m,5}$ and $H^{--m\hat{\mu}\pm}$ and, further, through the fundamental analytic vielbeins by Eqs. (102). For this purpose, we introduce the matrices

现在我们已经可以通过式 (102)，先将基本对象 $F, F^{\hat{\mu}}_{\alpha}$ 用负电荷标架 $H^{--m,5}$ 和 $H^{--m\hat{\mu}\pm}$ 表示，再进一步用基本解析标架表示。为此，我们引入矩阵

$$e^{\hat{\nu}}_{\hat{\mu}} := \partial^+_{\hat{\mu}} H^{--\hat{\nu}+}$$

$$e^{m,5}_{[\hat{\mu}\hat{\nu}]} := \partial^+_{\hat{\mu}}\partial^+_{\hat{\nu}} H^{--m,5} - \partial^+_{\hat{\mu}} e^{\hat{\rho}}_{\hat{\nu}} e^{-1\hat{\lambda}}_{\hat{\rho}} \partial^+_{\hat{\lambda}} H^{--m,5}$$

$$:= \left(e^{m,5}_{\mu\dot{\nu}}, e^{m,5}\varepsilon_{\mu\nu}, \bar{e}^{m,5}\varepsilon_{\dot{\mu}\dot{\nu}}\right). \tag{107}$$

The relation (106b) does not produce any new torsion constraints, since it is conventional and just serves to define the covariant vector derivative $\mathcal{D}_{\alpha\dot{\alpha}}$. The torsion constraints imposed by (106a) are as follows (the vanishing of the coefficients of ∂_m, ∂_5 and ∂^-_{μ} , respectively)

关系式 (106b) 不会产生新的挠率约束，因为它是约定性的，仅用于定义协变矢量导数 $\mathcal{D}_{\alpha\dot{\alpha}}$ 。(106a) 给出的挠率约束如下 (分别对应 ∂_m, ∂_5 和 ∂^-_{μ} 的系数为零)

$$E_\alpha^{\hat\mu} E_\beta^{\hat\upsilon} e^m_{[\hat\mu\hat\upsilon]} = 0, \tag{108}$$

$$E_\alpha^{\hat\mu} E_\beta^{\hat\upsilon} e^5_{[\hat\mu\hat\upsilon]} = \varepsilon_{\alpha\beta}, \tag{109}$$

$$E_\alpha^{\hat\mu} E_\beta^{\hat\upsilon} \partial^+_{\hat\mu} e^{\hat\lambda}_{\hat\upsilon} + \mathcal{D}^+_{[\alpha} E_{\beta]}^{\hat\rho} e^{\hat\lambda}_{\hat\rho} = 0 \tag{110}$$

(and their conjugates). In (110) we used the constraint (97a). One more constraint following from (106a) (the vanishing of the coefficient of $\partial^+_{\hat\mu}$) can easily be shown to be valid as a consequence of (110), while the latter itself proves also to be a consequence of the other ones. Note that under the superdiffeomorphism group (82) different torsions are linearly rotated through each other, so only the full set of the equations of the type (110) is covariant. As an example, let us give the transformation rule of the left-hand side of Eq. (110),

(以及它们的复共轭)。在 (110) 中我们使用了约束 (97a)。由 (106a) 可得到另一个约束 ($\partial^+_{\hat\mu}$ 的系数为零)，不难证明它是 (110) 的必然结果，而 (110) 本身也可由其他约束导出。需要注意，在超微分同胚群 (82) 下，不同挠率之间会发生线性旋转，因此只有这类方程 (110) 的完整集合才是协变的。举例来说，我们给出式 (110) 左侧的变换规则:

$$\mathcal{F}^{+\hat\lambda}_{\alpha\beta} \;:=\; E_\alpha^{\hat\mu} E_\beta^{\hat\upsilon} \partial^+_{\hat\mu} e^{\hat\lambda}_{\hat\upsilon} + \mathcal{D}^+_{[\alpha} E_{\beta]}^{\hat\rho} e^{\hat\lambda}_{\hat\rho}.$$

We have

我们有

$$\delta\mathcal{F}^{+\hat\lambda}_{\alpha\beta} = \Omega_\alpha{}^\gamma \mathcal{F}^{\hat\lambda}_{\gamma\beta} + \Omega_\beta{}^\gamma \mathcal{F}^{\hat\lambda}_{\alpha\gamma} + \Big[E_\alpha^{\hat\mu} E_\beta^{\hat\upsilon} e^m_{[\hat\mu\hat\upsilon]} \partial_m \lambda^{+\hat\lambda}$$

$$+ \mathcal{F}^{+\hat\gamma}_{\alpha\beta} \left(\partial^-_{\hat\gamma} \lambda^{+\hat\lambda} + e^{-1\hat\rho}_{\hat\gamma} \partial^+_{\hat\rho} H^{--m} \partial_m \lambda^{+\hat\lambda} \right) \Big].$$

We observe that $\mathcal{F}^{+\hat\lambda}_{\alpha\beta}$ homogeneously transform (as a tensor), provided that the constraint (108) is valid.

可以看到，当约束 (108) 成立时，$\mathcal{F}^{+\hat\lambda}_{\alpha\beta}$ 作齐次变换 (如同张量一样)。

The constraints serving to relate everything to the analytic potentials are just (108) and (109). In order to demonstrate this, we define, following [15],

将所有量与解析势关联的约束正是 (108) 和 (109)。为说明这一点，我们参照文献 [15] 定义:

$$f^{\mu\dot\upsilon} \;:=\; e^m e^{\mu\dot\upsilon}_m,\; e^m_{\rho\dot\lambda} e^{\mu\dot\upsilon}_m = \delta^\mu_\rho \delta^{\dot\upsilon}_{\dot\lambda},$$

$$f^5 \;:=\; e^5 - e^m e^{\mu\dot\upsilon}_m e^5_{\mu\dot\upsilon}. \tag{111}$$

Using (94), we now find that the constraint (108) amounts to the following equation

利用 (94)，我们现在可得约束 (108) 等价于如下方程

$$F^{\dot{\upsilon}}_{\alpha} = f^{\dot{\upsilon}}_{\alpha} + A\bar{f}^{\dot{\upsilon}}_{\alpha}, \; A \;:=\; \frac{1}{2}F^{\dot{\upsilon}\alpha}F_{\dot{\upsilon}\alpha}. \tag{112}$$

Squaring this equation, we obtain the quadratic equation for determining A :

将该方程平方后，我们得到用于确定 A 的二次方程:

$$A^2\bar{f}^2 + 2A\left(f\bar{f} - 1\right) + f^2 = 0, \tag{113}$$

$$f^2 \;:=\; f^{\dot{\upsilon}\alpha}f_{\dot{\upsilon}\alpha}, \bar{f}^2 \;:=\; \bar{f}^{\dot{\upsilon}\alpha}\bar{f}_{\dot{\upsilon}\alpha} = \overline{(f^2)}, f\bar{f} \;:=\; f^{\dot{\upsilon}\alpha}\bar{f}_{\dot{\upsilon}\alpha}.$$

A non-singular solution of (113) is

(113) 的一个非奇异解为

$$A = f^2\left[1 - f\bar{f} + \sqrt{\left(1 - f\bar{f}\right)^2 - f^2\bar{f}^2}\right]^{-1}. \tag{114}$$

Analogously, Eq. (109) is reduced to

同理，方程 (109) 可化简为

$$F^{-2} = e^5 + A\bar{e}^5 - F^{\mu\dot{\upsilon}}e^5_{\mu\dot{\upsilon}},$$

whence, after using Eqs. (112) and (111),

由此，代入方程 (112) 和 (111) 后可得

$$F = \frac{1}{\sqrt{f^5 + A\bar{f}^5}}. \tag{115}$$

Building Blocks and Invariant Action

构造块与不变作用量

So far, we have succeeded to relate the components of (94) to the quantities $e^{m,5}_{\mu\dot{\upsilon}}, e^{m,5}, \bar{e}^{m,5}$ which are expressed through the negatively charged vielbeins by Eqs. (107) and, further, through the basic analytic vielbeins H^{++M} , using the harmonic flatness conditions (102). Note that the relations (112), (114) and (115) were derived in [15] without directly solving the torsion constraints (108), (109), based only upon the guessed transformation properties of the involved quantities. In our presentation, all these transformation properties can be consistently deduced from the explicit form of the matrix entries (107) and, further, from the definitions (111). The basic transformation laws from which all others can be derived are the following

截至目前，我们已经成功将式 (94) 的分量与 $e^{m,5}_{\mu\hat{\upsilon}}, e^{m,5}, \bar{e}^{m,5}$ 这些量联系起来，这些量由式 (107) 通过带负电荷的标架场表示，进一步还可利用调和平坦条件 (102) 通过基本解析标架场 H^{++M} 表示。注意，关系式 (112)、(114) 和 (115) 是文献 [15] 在不直接求解挠率约束 (108)、(109) 的情况下推导得到的，推导仅基于对相关量变换性质的猜测。在本文的表述中，所有这些变换性质都可以从矩阵元的显式形式 (107)，再进一步从定义 (111) 中一致推导得出。所有其他变换规律都可由下述基本变换规律导出

$$\delta e^{\hat{\rho}}_{\hat{\mu}} = -\partial^{+}_{\hat{\mu}}\lambda^{-\hat{\upsilon}} e^{\hat{\rho}}_{\hat{\upsilon}} + e^{\hat{\omega}}_{\hat{\mu}}\partial^{-}_{\hat{\omega}}\lambda^{+\hat{\rho}} + \partial^{+}_{\hat{\mu}} H^{--m}\partial_m\lambda^{+\hat{\rho}},$$

$$\delta e^{m,5}_{[\hat{\mu}\hat{\upsilon}]} = e^{n}_{[\hat{\mu}\hat{\upsilon}]}\lambda^{m,5}_{n} - \partial^{+}_{\hat{\mu}}\lambda^{-\hat{\omega}} e^{m,5}_{[\hat{\omega}\hat{\upsilon}]}, +\partial^{+}_{\hat{\upsilon}}\lambda^{-\hat{\omega}} e^{m,5}_{[\hat{\omega}\hat{\mu}]}, \tag{116}$$

where

其中

$$\lambda^{m,5}_{n} := \partial_n\lambda^{m,5} - \partial_n\lambda^{+\hat{\rho}} e^{-1\hat{\gamma}}_{\hat{\rho}}\partial^{+}_{\hat{\gamma}} H^{--m,5}. \tag{117}$$

The basic use of the quantities constructed above is related to the possibility to construct the invariant volume of the curved harmonic superspace. In the analytic parametrization, it is defined as

上述构造量的基本用途与构造弯曲调和超空间不变体积的可能性相关。在解析参数化下，其定义为

$$\mu(Z) = dud^{12}ZE^{-1} \equiv dud^4x_A d^4\theta^{+}_{A} d^4\theta^{-}_{A} E^{-1},\ E := \mathrm{Ber}\, E^{M}_{A},$$

$$\delta\left(dud^{12}Z\right) = dud^{12}Z\left(\partial_m\lambda^m - \partial^{+}_{\hat{\upsilon}}\lambda^{-\hat{\upsilon}} - \partial^{-}_{\hat{\upsilon}}\lambda^{+\hat{\upsilon}}\right),$$

$$\delta E = \left(\partial_m\lambda^m - \partial^{+}_{\hat{\upsilon}}\lambda^{-\hat{\upsilon}} - \partial^{-}_{\hat{\upsilon}}\lambda^{+\hat{\upsilon}}\right)E, \tag{118}$$

where E^M_A is a 12×12 matrix supervielbein entering the spinor and vector covariant derivatives (80),(104) and (105). Instead of calculating the Berezinian E directly, it is much simpler to construct this object from requiring it to have the transformation property (118). The sought superdensity is given by the following expression

其中 E^M_A 是进入旋量和矢量协变导数 (80)、(104) 和 (105) 的 12×12 矩阵超标架场。无需直接计算贝雷辛 ian E，从变换性质要求 (118) 出发构造该对象要简单得多。我们所需的超密度由下式给出

$$E = \det\left(e^{m}_{\mu\dot{\mu}}\right)\overset{-1}{\det}\left(e^{\hat{\mu}}_{\hat{\upsilon}}\right)\sqrt{\left(1 - f\bar{f}\right)^2 - f^2\bar{f}^2}. \tag{119}$$

Using the transformation laws (116) and their consequences, it is straightforward to check that the expression (119) indeed transforms according to (118).

利用变换规律 (116) 及其推论，可以直接验证式 (119) 确实按照 (118) 变换。

It was shown in [15] that the invariant superfield action of the given version of $\mathcal{N}=2$SG is given by the expression

文献 [15] 已证明，该版本 $\mathcal{N}=2$SG 的不变超场作用量由下式给出

$$S^{\mathcal{N}=2}_{SG} = -\frac{1}{\kappa^2}\int du d^4x_A d^4\theta^+_A d^4\theta^-_A E^{-1} H^{++5} H^{--5}. \tag{120}$$

Its invariance under the local shifts of $x^m_A, \theta^{\pm\hat{\mu}}_A$ is evident, as H^{++5}, H^{--5} are scalars with respect to these transformations. To prove its invariance under gauge transformations with the analytic parameter λ^5 which do not act on the coordinates in (120) is a much more involved technical task. We invite the interested reader to consult the original paper [15] and the book [14]. Here we will limit our attention to the linearized version of this action following Ref. [16] and a recent paper [17].

它在 $x^m_A, \theta^{\pm\hat{\mu}}_A$ 局域平移下的不变性是显然的，因为 H^{++5}, H^{--5} 相对于这些变换是标量。要证明它在携带解析参数 λ^5 的规范变换下 (该变换不作用于 (120) 中的坐标) 的不变性，则是一项复杂得多的技术任务。感兴趣的读者可以查阅原始文献 [15] 和专著 [14]。本文我们将仿照文献 [16] 和近期论文 [17]，仅关注该作用量的线性化版本。

Linearized Approximation

线性化近似

The first step of the linearization consists in singling out the θ dependent backgrounds in the analytic potentials

线性化的第一步是在解析势中分离出依赖 θ 的背景

$$H^{++\alpha\dot{\beta}} = -2i\theta^{+\alpha}\bar{\theta}^{+\dot{\beta}} + h^{++\alpha\dot{\beta}},$$

$$H^{++5} = i\left(\theta^{\hat{+}}\right)^2 + h^{++5},\ \left(\theta^{\hat{+}}\right)^2 := \left(\theta^{+\alpha}\theta^+_\alpha - \bar{\theta}^+_{\dot{\alpha}}\bar{\theta}^{+\dot{\alpha}}\right),$$

$$H^{++\hat{\mu}+} = h^{++\hat{\mu}+} \tag{121}$$

The negatively charged "shifted" potentials $h^{--\alpha\dot{\beta},5}$ are singled out from $H^{--\alpha\dot{\beta},5}$ by the analogous relations, with the background pieces $-2i\theta^{-\alpha}\bar{\theta}^{-\dot{\beta}}$ and $i(\theta^-)^2$, respectively. Both sets of potentials are interrelated by the flatness conditions following from the linearization of Eqs. (102):

带负电的“偏移”势 $h^{--\alpha\dot{\beta},5}$ 通过类似关系从 $H^{--\alpha\dot{\beta},5}$ 中分离出来，背景项分别为 $-2i\theta^{-\alpha}\bar{\theta}^{-\dot{\beta}}$ 和 $i(\theta^-)^2$ 。这两组势由式 (102) 线性化后得到的平坦性条件关联:

$$D^{++}h^{--\alpha\dot{\alpha}} - D^{--}h^{++\alpha\dot{\alpha}} + 2i\left(h^{--\alpha+}\bar{\theta}^{+\dot{\alpha}} + \theta^{+\alpha}h^{--\dot{\alpha}+}\right) = 0,$$

$$D^{++}h^{--5} - D^{--}h^{++5} - 2i\left(h^{--\alpha+}\theta^{+}_{\alpha} - \bar{\theta}^{+}_{\dot\alpha}h^{--\dot\alpha+}\right) = 0, \tag{122}$$

$$D^{++}h^{--\alpha+} - D^{--}h^{++\alpha+} = 0,\ D^{++}h^{--\dot\alpha+} - D^{--}h^{++\dot\alpha+} = 0,$$

$$D^{++}h^{--\alpha-} - h^{--\alpha+} = 0,\ D^{++}h^{--\dot\alpha-} - h^{--\dot\alpha+} = 0. \tag{123}$$

These constraints are invariant under the following linearized form of the superfield gauge transformations defined in (88) and (101)

这些约束在式 (88) 和 (101) 定义的超场规范变换的如下线性化形式下保持不变

$$\delta_\lambda h^{\pm\pm\alpha\dot\alpha} = D^{\pm\pm}\lambda^{\alpha\dot\alpha} + 2i\left(\lambda^{\pm\alpha}\bar{\theta}^{\pm\dot\alpha} + \theta^{+\alpha}\bar{\lambda}^{+\dot\alpha}\right)$$

$$\delta_\lambda h^{\pm\pm5} = D^{\pm\pm}\lambda^{5} - 2i\left(\lambda^{\pm\alpha}\theta^{\pm}_{\alpha} - \bar{\theta}^{\pm}_{\dot\alpha}\bar{\lambda}^{\pm\dot\alpha}\right), \tag{124}$$

$$\delta_\lambda h^{++\hat\alpha+} = D^{++}\lambda^{+\hat\alpha}, \tag{125}$$

$$\delta_\lambda h^{--\hat\alpha+} = D^{--}\lambda^{+\hat\alpha} - \lambda^{-\hat\alpha}, \tag{126}$$

$$\delta_\lambda h^{--\hat\alpha-} = D^{--}\lambda^{-\hat\alpha}. \tag{127}$$

In all these formulas,

在所有这些公式中,

$$D^{\pm\pm} = \partial^{\pm\pm} - 2i\theta^{\pm\alpha}\bar{\theta}^{\pm\dot\alpha}\partial_{\alpha\dot\beta} + \theta^{\pm\hat\mu}\frac{\partial}{\partial\theta^{\mp\hat\mu}},\ \partial_{\alpha\dot\beta} := \sigma^{m}_{\alpha\dot\beta}\partial_m. \tag{128}$$

A peculiarity of the linearization limit considered is the unusual realization of the flat $\mathcal{N}=2$ supersymmetry on the potentials $h^{\pm\pm m,5}$ (We denote by δ_ε so called passive transformations which differ from the more accustomed ”active” transformations δ^*_ε by the ”transport term”, $\delta^*_\varepsilon = \delta_\varepsilon - \delta_\varepsilon Z^M\partial_M$.)

我们考虑的线性化极限的一个特殊之处在于, 平坦 $\mathcal{N}=2$ 超对称在势 $h^{\pm\pm m,5}$ 上存在非寻常实现 (我们将 δ_ε 称为被动变换,它与我们更熟悉的“主动”变换 δ^*_ε 的区别在于存在“传输项”$\delta^*_\varepsilon = \delta_\varepsilon - \delta_\varepsilon Z^M\partial_M$ 。)

$$\delta_\varepsilon h^{\pm\pm\alpha\dot\alpha} = -2i\left(h^{\pm\pm\alpha+}\bar{\varepsilon}^{-\dot\alpha} + \varepsilon^{-\alpha}h^{\pm\pm\dot\alpha+}\right),\ \delta_\varepsilon h^{\pm\pm5} = 2ih^{\pm\hat\mu+}\varepsilon^{-}_{\hat\mu}.$$

$$\delta_\varepsilon h^{++\hat\mu+} = \delta_\varepsilon h^{--\hat\mu+} = \delta h^{--\hat\mu-} = 0. \tag{129}$$

It is straightforward to be convinced that Eqs. (124)-(127) are covariant just under such modified transformation laws. Actually, these transformation laws are valid in the full nonlinear case too. The difference between the nonlinear and linearized cases is that in the former case these rigid transformations form a subgroup of the gauge group (82) (and its counterpart for the negatively charged vielbeins), while in the latter case, they constitute an independent symmetry (which form a semi-direct product with the relevant linearized gauge transformations (126) and (127)).

不难发现，式 (124)-(127) 恰好在这类修正变换规律下是协变的。实际上这类变换规律在完全非线性情况下同样成立。非线性情形和线性化情形的区别在于: 前者中这些刚性变换构成规范群 (82)(及其对应负电标架的对应群) 的一个子群，而后者中这些刚性变换构成独立对称性，它与相关线性化规范变换 (126)、(127) 构成半直积)

The next step in constructing the linearized invariant action is to define the important non-analytic superfields

构造线性化不变作用量的下一步是定义重要的非解析超场

$$G^{\pm\pm\alpha\dot{\alpha}} \ := \ h^{\pm\pm\alpha\dot{\alpha}} + 2i\left(h^{\pm\pm\alpha+}\bar{\theta}^{-\dot{\alpha}} + \theta^{-\alpha}h^{\pm\pm\dot{\alpha}+}\right), \tag{130}$$

$$G^{\pm\pm5} \ := \ h^{\pm\pm5} - 2ih^{\pm\pm\hat{\alpha}+}\theta^{-}_{\hat{\alpha}}. \tag{131}$$

It is easy to check that the newly defined objects transform as the standard scalar $\mathcal{N}=2$ superfields

容易验证，新定义的对象变换为标准标量 $\mathcal{N}=2$ 超场

$$\delta_\varepsilon G^{\pm\pm\alpha\dot{\alpha}} = \delta_\varepsilon G^{\pm\pm5} = 0. \tag{132}$$

They also display simple transformation properties under the gauge transformations (125)-(127)

它们在规范变换 (125)-(127) 下也具有简单的变换性质

$$\delta_\lambda G^{\pm\pm\alpha\dot{\alpha}} = D^{\pm\pm}\Lambda^{\alpha\dot{\alpha}}, \ \delta_\lambda G^{\pm\pm5} = D^{\pm\pm}\Lambda^5, \tag{133}$$

$$\Lambda^{\alpha\dot{\alpha}} = \lambda^{\alpha\dot{\alpha}} + 2i\left(\lambda^{+\alpha}\bar{\theta}^{-\dot{\alpha}} + \theta^{-\alpha}\bar{\lambda}^{+\dot{\alpha}}\right), \ \Lambda^5 = \lambda^5 - 2i\lambda^{+\hat{\alpha}}\theta^-_{\hat{\alpha}}, \tag{134}$$

and satisfy the harmonic flatness conditions

并且满足调和平坦性条件

$$D^{++}G^{--\alpha\dot{\alpha}} = D^{--}G^{++\alpha\dot{\alpha}}, \ D^{++}G^{--5} = D^{--}G^{++5} \tag{135}$$

as a direct consequence of the harmonic equations (122)-(123). The invariant linearized action of $\mathcal{N}=2$ SG can be constructed just from these objects.

这是调和方程 (122)-(123) 的直接推论。$\mathcal{N}=2$ 超引力的不变线性化作用量恰好可以由这些对象构造。

Let us consider the manifestly $\mathcal{N}=2$ supersymmetric ”trial” action

我们来看明显满足 $\mathcal{N}=2$ 超对称的“试探”作用量

$$S_1 = \int dud^4xd^8\theta G^{++\alpha\dot{\alpha}} G^{--}_{\alpha\dot{\alpha}}. \tag{136}$$

Its gauge variation, with taking into account the relation (135), can be transformed to the expression

考虑到关系式 (135)，它的规范变分可以变换为如下表达式

$$\delta_\lambda S_1 = \frac{1}{2} \int dud^4xd^8\theta D^{--} \Lambda^{\alpha\dot{\alpha}} G^{++}_{\alpha\dot{\alpha}}. \tag{137}$$

Now we pass to the integral over the analytic subspace,

现在我们对解析子空间积分，

$$\int dud^4xd^8\theta = \int dud\zeta^{(-4)} (D^+)^4, \ (D^+)^4 = \frac{1}{16} (\bar{D}^+)^2 (D^+)^2, \tag{138}$$

and, after some algebra, using the property that both $\Lambda^{\alpha\dot{\alpha}}$ and $G^{++}_{\alpha\dot{\alpha}}$ are linear in $\theta^-_\alpha, \overline{\theta}^-_{\dot{\beta}}$ with analytic coefficients, reduce this variation to

经过若干代数运算，利用 $\Lambda^{\alpha\dot{\alpha}}$ 和 $G^{++}_{\alpha\dot{\alpha}}$ 都对 $\theta^-_\alpha, \overline{\theta}^-_{\dot{\beta}}$ 线性且系数为解析的性质，可将该变分化简为

$$\delta_\lambda S_1 = 2i \int dud\zeta^{(-4)} \left(\partial_{\beta\dot{\beta}} \lambda^{+\beta} h^{++\dot{\beta}+} - \partial_{\beta\dot{\beta}} \overline{\lambda}^{+\dot{\beta}} h^{++\beta+} \right). \tag{139}$$

As the next step, we define

下一步，我们定义

$$S_2 = \int dud^4xd^8\theta G^{++5} G^{--5}. \tag{140}$$

Applying similar manipulations, we find

应用类似操作，我们得到

$$\delta_\lambda S_2 = -2i \int dud\zeta^{(-4)} \left(\partial_{\beta\dot{\beta}} \lambda^{+\beta} h^{++\dot{\beta}+} - \partial_{\beta\dot{\beta}} \overline{\lambda}^{+\dot{\beta}} h^{++\beta+} \right). \tag{141}$$

So the sum

因此求和式

$$S_{(s=2)} \sim -(S_1 + S_2) = -\frac{1}{\kappa^2} \int dud^4xd^8\theta \left(G^{++\alpha\dot{\alpha}} G^{--}_{\alpha\dot{\alpha}} + G^{++5} G^{--5} \right) \tag{142}$$

is invariant under both rigid $\mathcal{N}=2$ supersymmetry and linearized gauge transformations (note the sign minus which is characteristic of the compensator actions). It is the invariant action of the linearized $\mathcal{N}=2$SG and the true $\mathcal{N}=2$ extension of the free spin 2 action. In the HSS approach it was firstly given (in a different form) in [16].

在刚性 $\mathcal{N}=2$ 超对称与线性化规范变换下均不变(注意补偿子作用特有的负号)。它是线性化 $\mathcal{N}=2$SG 的不变作用量，也是自由自旋 2 作用量的真实 $\mathcal{N}=2$ 延拓。在 HSS 方法中，它最早以不同形式出现在文献 [16] 中。

Note that the linearized $\mathcal{N}=2$SG in the ordinary $\mathcal{N}=2$ superspace was considered in [56,57] and, more recently, e.g., in [58]. There, the approach based on the Mezincescu-type prepotentials was applied and some other versions of $\mathcal{N}=2$ SG were also considered. The present derivation uses the objects constructed out of the fundamental analytic gauge potentials.

注意，普通 $\mathcal{N}=2$ 超空间中的线性化 $\mathcal{N}=2$SG 已在文献 [56,57] 中被研究，更近的研究例如文献 [58]。这些研究采用了基于 Mezincescu 型预备势的方法，也讨论了 $\mathcal{N}=2$ 超引力的其他版本。本文推导使用的对象由基本解析规范势构造而来。

Finally, we briefly discuss how the correct component action for the spin 2 can be deduced from (142). The purely gravity parts of the potentials $G^{++\alpha\dot{\alpha}}$ and G^{++5} , in WZ gauge are given by the expressions

最后，我们简要讨论如何从式 (142) 推导出正确的自旋 2 分量作用量。在 WZ 规范下，势 $G^{++\alpha\dot{\alpha}}$ 和 G^{++5} 的纯引力部分由下式给出

$$G^{++\alpha\dot{\alpha}}(h) = -2i\kappa\theta^{+\beta}\bar{\theta}^{+\dot{\beta}} h^{\alpha\dot{\alpha}}_{\beta\dot{\beta}} + 4(\theta^{+})^{2}\bar{\theta}^{+\dot{\beta}}\bar{\theta}^{-\dot{\alpha}} B^{\alpha}_{\dot{\beta}} - 4\left(\bar{\theta}^{+}\right)^{2}\theta^{+\beta}\theta^{-\alpha}B^{\dot{\alpha}}_{\beta},$$

$$G^{++5}(h) = -4(\theta^{+})^{2}\bar{\theta}^{+\dot{\rho}}\theta^{-}_{\mu}B^{\mu}_{\dot{\rho}} - 4\left(\bar{\theta}^{+}\right)^{2}\theta^{+\mu}\bar{\theta}^{-}_{\dot{\rho}}\bar{B}^{\dot{\rho}}_{\mu}, \tag{143}$$

where

其中

$$B_{\mu\dot{\mu}} = \frac{\kappa}{4}\left(3\partial_{\mu\dot{\mu}}h - \partial^{\rho\dot{\rho}}h_{(\mu\rho)(\dot{\mu}\dot{\rho})}\right), \; h_{\alpha\beta\dot{\alpha}\dot{\beta}} = h_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}h \tag{144}$$

and we gauged away the rest of components of $h_{\alpha\beta\dot{\alpha}\dot{\beta}}$, i.e., $\varepsilon_{\alpha\beta}h_{(\dot{\alpha}\dot{\beta})}, \varepsilon_{\dot{\alpha}\dot{\beta}}h_{(\alpha\beta)}$ by the linearized local "Lorentz" transformations. After solving the relevant parts of the zero-curvature conditions (135) for $G^{--\alpha\dot{\alpha}}(h)$, $G^{--5}(h)$, substituting all that in the action (142) and doing there the θ integration, we obtain

我们通过线性化局域“洛伦兹”变换规范固定消去了 $h_{\alpha\beta\dot{\alpha}\dot{\beta}}$ 即 $\varepsilon_{\alpha\beta}h_{(\dot{\alpha}\dot{\beta})}, \varepsilon_{\dot{\alpha}\dot{\beta}}h_{(\alpha\beta)}$ 的其余分量。对 $G^{--\alpha\dot{\alpha}}(h)$, $G^{--5}(h)$ 求解零曲率条件 (135) 的相关部分，将所有结果代入作用量 (142) 并完成 θ 积分后，我们得到

$$S^{(s=2)}_{(h)} = -\int d^4x \left[h^{(\alpha\beta)(\dot{\alpha}\dot{\beta})}\Box h_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} - h^{(\alpha\beta)(\dot{\alpha}\dot{\beta})}\partial_{\alpha\dot{\alpha}}\partial^{\rho\dot{\rho}}h_{(\rho\beta)(\dot{\rho}\dot{\beta})}\right.$$

$$+2h\partial^{\alpha\dot\alpha}\partial^{\beta\dot\beta}h_{(\alpha\beta)(\dot\alpha\dot\beta)}-6h\Box h\Big]. \tag{145}$$

It is the correct free action of the spin 2 fields $\left(h_{(\alpha\beta)(\dot\alpha\dot\beta)},h\right)$. It is invariant under the corresponding gauge transformations,

这就是自旋 2 场 $\left(h_{(\alpha\beta)(\dot\alpha\dot\beta)},h\right)$ 的正确自由作用量，它在对应的规范变换下不变，

$$\delta h_{(\alpha\beta)(\dot\alpha\dot\beta)}=\frac{1}{\kappa}\partial_{(\alpha(\dot\alpha}a_{\beta\dot\beta)},\ \delta h=\frac{1}{4\kappa}\partial_{\alpha\dot\alpha}a^{\alpha\dot\alpha}, \tag{146}$$

where $a^{\alpha\dot\alpha}$ is a vector parameter of the linearized diffeomorphisms. For the spin 1 gauge field of the $\mathcal{N}=2$ spin2multiplet the action (142) also gives rise to the correct free Maxwell action (The correct sign in this case is restored after elimination of the symmetric tensor auxiliary fields $T^{(\alpha\beta)},\bar T^{(\dot\alpha\dot\beta)}$ which are present in $G^{\pm\pm\alpha\dot\alpha},G^{\pm\pm5}$ and have the same dimension as the Maxwell field strength [17].).

其中 $a^{\alpha\dot\alpha}$ 是线性化微分同胚的矢量参数。对于 $\mathcal{N}=2$ 自旋 2 多重态的自旋 1 规范场，作用量 (142) 同样可以给出正确的自由麦克斯韦作用量 (消去 $G^{\pm\pm\alpha\dot\alpha},G^{\pm\pm5}$ 中存在的、量纲与麦克斯韦场强相同的对称张量辅助场 $T^{(\alpha\beta)},\bar T^{(\dot\alpha\dot\beta)}$ 后，即可恢复正确的符号 [17])。

Conformal $\mathcal{N}=2$ Supergravity

共形 $\mathcal{N}=2$ 超引力

Like in the $\mathcal{N}=1$ case, the basic object of various versions of Einstein $\mathcal{N}=2$ supergravity is the irreducible multiplet of conformal $\mathcal{N}=2$ supergravity - the superspin 1, superisospin 0$\mathcal{N}=2$ Weyl multiplet. Any version of Einstein $\mathcal{N}=2$ supergravity can be obtained as a theory of the appropriate matter compensator in the background of this $\mathcal{N}=2$ Weyl multiplet [59].

与 $\mathcal{N}=1$ 情形类似，各类爱因斯坦 $\mathcal{N}=2$ 超引力的基础对象是共形 $\mathcal{N}=2$ 超引力的不可约多重态——超自旋 1、超同位旋 0$\mathcal{N}=2$ 外尔多重态。任何版本的爱因斯坦 $\mathcal{N}=2$ 超引力都可以视作该 $\mathcal{N}=2$ 外尔多重态背景下，相应物质补偿场的理论 [59]。

$\mathcal{N}=2$ Weyl Multiplet in HSS

$\mathcal{N}=2$ 魏尔多重态 (调和超空间中)

In a nice analogy with $\mathcal{N}=1$ conformal supergravity (see section "Harmonic Superspace: The Definition"), in the HSS approach the $\mathcal{N}=2$ Weyl multiplet is closely related to the fundamental group of conformal $\mathcal{N}=2$ supergravity and has a clear geometrical meaning. The underlying group is the group of general diffeomorphisms of the harmonic analytic superspace $\left(\zeta^M,u^{\pm i}\right)$, while the Weyl multiplet is accommodated by the harmonic analytic vielbeins, which covariantize the analyticity-preserving harmonic derivative D^{++} with respect to this group. So the group-theoretical basis of $\mathcal{N}=2$ supergravity is the preservation of $\mathcal{N}=2$ harmonic analyticity in the curved case, much like the preservation of $\mathcal{N}=1$ chirality provides the basis of

$\mathcal{N} = 1$ supergravity. An essential difference from the group of Einstein SG (82),(83) is the presence of local $SU(2)$ symmetry and the corresponding nontrivial transformations of the harmonic variables:

> 通过与 $\mathcal{N} = 1$ 共形超引力的恰当类比 (参见章节“调和超空间: 定义”)，在 HSS 方法中，$\mathcal{N} = 2$ 魏尔多重态与共形 $\mathcal{N} = 2$ 超引力的基本群密切相关，具有清晰的几何意义。其 underlying 群是调和解析超空间 $(\zeta^M, u^{\pm i})$ 的一般微分同胚群，而魏尔多重态由调和解析标架承载，标架将保持解析性的调和导数 D^{++} 相对于该群协变化。因此 $\mathcal{N} = 2$ 超引力的群论基础是在弯曲情形下保持 $\mathcal{N} = 2$ 调和解析性，这非常类似于保持 $\mathcal{N} = 1$ 手征性构成了 $\mathcal{N} = 1$ 超引力的基础。它与爱因斯坦超引力 (82),(83) 对应的群的一个本质区别，是存在局域 $SU(2)$ 对称性以及调和变量对应的非平凡变换:

$$\delta x^m = \lambda^m(\zeta, u)$$

$$\delta \theta^{+\hat{\mu}} = \lambda^{+\hat{\mu}}(\zeta, u)$$

$$\delta \theta^{-\hat{\mu}} = \lambda^{-\hat{\mu}}\left(\zeta, \theta^{-\hat{\mu}}, u\right)$$

$$\delta u_i^+ = \lambda^{++}(\zeta, u)\, u_i^-, \ \delta u_i^- = 0, \tag{147}$$

whereas before $\hat{\mu} := (\mu, \dot{\mu})$. The local parameters λ in (147), except for $\lambda^{-\hat{\mu}}\left(\zeta, \theta^{-\hat{\mu}}, u\right)$ are arbitrary analytic harmonic functions. Note that only the harmonics u^+ but not u^- transform, yet preserving the harmonic defining relation $u^{+i}u_i^- = 1$. This peculiarity is related to the special realization of the rigid $\mathcal{N} = 2$ superconformal group $SU(2,2 \mid 2)$ in the analytic superspace (see [14]). The transformations (147) provide gauging of this rigid group. Using the standard Lie-bracket formalism, it is straightforward to make sure that the transformations (147) indeed form a group. In particular,

> 这和之前 $\hat{\mu} := (\mu, \dot{\mu})$ 的情况不同。(147) 中的局域参数 λ 除 $\lambda^{-\hat{\mu}}\left(\zeta, \theta^{-\hat{\mu}}, u\right)$ 外都是任意解析调和函数。注意只有调和量 u^+ 发生变换，u^- 不变换，且仍满足调和定义关系 $u^{+i}u_i^- = 1$。这一特殊性与刚性 $\mathcal{N} = 2$ 超共形群 $SU(2,2 \mid 2)$ 在解析超空间中的特殊实现有关 (参见文献 [14])。变换 (147) 对该刚性群实现了规范作用。利用标准李括号形式体系，不难验证变换 (147) 确实构成一个群，特别地:

$$[\delta_1, \delta_2]\, u_i^+ = \left\{\left(\lambda_1^m \partial_m + \lambda_1^{+\hat{\mu}} \partial_{\hat{\mu}}^- + \lambda_1^{++} \partial^{--}\right) \lambda_2^{++}\right\} u_i^- - (1 \leftrightarrow 2)\,. \tag{148}$$

We see that the commutator produces a transformation of u_i^+ of the same type as in (147), with a new analytic parameter.

> 我们可以看到，对易子生成的 u_i^+ 变换与 (147) 的类型相同，只是对应一个新的解析参数。

The starting point of the actual construction of conformal $\mathcal{N} = 2$ supergravity is the conformally invariant free action of the hypermultiplet in the flat HSS [14]:

> 构造共形 $\mathcal{N} = 2$ 超引力的实际起点，是平坦 HSS 中超多重态的共形不变自由作用量 [14]:

$$S = \frac{1}{2}\int dud\zeta^{(-4)} q^{+a} D^{++} q_a^+,\ a = 1, 2. \tag{149}$$

Note the "wrong" sign of this action. The rigid conformal supergroup $SU(2,2 \mid 2)$ leaves the analytic superspace (ζ_A, u) invariant (see [14]), while D^{++} has an unusual transformation law

注意该作用量的符号是“反常”的。刚性共形超群 $SU(2,2 \mid 2)$ 保持解析超空间 (ζ_A, u) 不变 (参见文献 [14])，而 D^{++} 遵循非平凡的变换规律

$$\delta D^{++} = -\lambda_{rig}^{++} D^0, \tag{150}$$

where λ_{rig}^{++} is the "flat superspace" limit of the gauge super parameter λ^{++} in (147). The lagrangian in (149) is invariant under (150) just because of the algebraic property $q^{+a} q_a^+ = 0$. The integration measure in (149) is not invariant, but its transformation can be compensated by the appropriate coordinate-dependent rescaling of q^{+a} , once again exploiting the identical vanishing of $q^{+a} q_a^+$. The details related to the analytic superspace preserving realization of the supergroup $SU(2,2 \mid 2)$ can be found in [14].

其中 λ_{rig}^{++} 是 (147) 中规范超参数 λ^{++} 的“平坦超空间”极限。由于代数性质 $q^{+a} q_a^+ = 0$，(149) 中的拉格朗日量在 (150) 变换下保持不变。(149) 中的积分测度本身并不不变，但利用 $q^{+a} q_a^+$ 的恒等于零这一性质，可通过对 q^{+a} 做依赖于坐标的适当标度变换补偿测度的变换。关于保持解析超空间的超群 $SU(2,2 \mid 2)$ 实现的更多细节可以参见文献 [14]。

The covariantization of the action (149) amounts to covariantization of the flat harmonic derivative

作用量 (149) 的协变化本质上就是对平坦调和导数做协变化

$$D^{++} \to \mathcal{D}^{++}$$

$$\mathcal{D}^{++} = \partial^{++} + H^{+4}\partial^{--} + H^{++m}\partial_m + H^{++\hat{\mu}+}\partial_{\hat{\mu}}^- + H^{++\hat{\mu}-}\partial_{\hat{\mu}}^+, \tag{151}$$

where all the coefficients of the supervielbein, except for $H^{++\hat{\mu}-}$, are analytic superfields, $H^{++M} = H^{++M}(\zeta, u), H^{+4} = H^{+4}(\zeta, u), M := (m, \hat{\mu}+)$. This analyticity is necessary in order to preserve the analyticity of the Lagrangian in (149). Though $H^{++\hat{\mu}-}$ drops out from the covariantized q^+ action because of analyticity of the superfield q^+ , it should in general be included, having in mind that $\mathcal{D}^{++}$ should be as well covariant when acting on non-analytic harmonic superfields. The natural generalization of the flat transformation law (150),

其中除 $H^{++\hat{\mu}-}$ 外，所有超 vielbein 的系数都是解析超场 $H^{++M} = H^{++M}(\zeta, u), H^{+4} = H^{+4}(\zeta, u), M := (m, \hat{\mu}+)$。该解析性是保证 (149) 中拉格朗日量解析性的必要条件。尽管由于超场 q^+ 的解析性，$H^{++\hat{\mu}-}$ 会从协变 q^+ 作用量中消去，但一般情况下仍需保留它，因为作用于非解析调和超场时，$\mathcal{D}^{++}$ 也应当是协变的。平直变换规律 (150) 的自然推广，

$$\delta\mathcal{D}^{++} = -\lambda^{++} D^0 \tag{152}$$

implies the following transformation properties for the supevielbeins

给出了超 vielbein 满足的如下变换性质

$$\delta H^{++M} = \mathcal{D}^{++}\lambda^{M} - \delta^{M}_{+\hat{\mu}}\theta^{+\hat{\mu}}\lambda^{++},\ \delta H^{++\hat{\mu}-} = \mathcal{D}^{++}\lambda^{-\hat{\mu}} + \theta^{-\hat{\mu}}\lambda^{++},$$

$$\delta H^{+4} = \mathcal{D}^{++}\lambda^{++}, \tag{153}$$

$$D^{0} \equiv \partial^{0} + \theta^{+\hat{\mu}}\partial^{-}_{\hat{\mu}} - \theta^{-\hat{\mu}}\partial^{+}_{\hat{\mu}}.$$

The non-analytic gauge superparameter $\lambda^{-\hat{\mu}}\left(\zeta, \theta^{-\hat{\mu}}, u\right)$ contains enough component parameters in order to gauge the non-analytic vielbein coefficient $H^{++\hat{\mu}-}$ into its flat limit,

非解析规范超参数 $\lambda^{-\hat{\mu}}\left(\zeta, \theta^{-\hat{\mu}}, u\right)$ 包含足够的分量参数，可通过规范变换将非解析 vielbein 系数 $H^{++\hat{\mu}-}$ 固定到其平直极限,

$$H^{++\hat{\mu}-} = \theta^{+\hat{\mu}},\ \Rightarrow\ \mathcal{D}^{++}\lambda^{-\hat{\mu}} = \lambda^{+\hat{\mu}} - \theta^{-\hat{\mu}}\lambda^{++}. \tag{154}$$

The covariantized hypermultiplet action,

协变超多重态作用量,

$$S_{\text{cov}} = \int du d\zeta^{(-4)} q^{+a}\mathcal{D}^{++}q^{+}_{a},\ a = 1, 2, \tag{155}$$

is invariant under (147), (152) by the same token as in the flat case under supercon-formal group, just due to the property $q^{+a}q^{+}_{a} = 0$. Indeed, the transformation (152) leaves the action invariant because of the evident property $D^{0}q^{+a} = q^{+a}$, while the transformation of the analytic integration measure,

与平直情形下超共形群的情况同理，仅由性质 $q^{+a}q^{+}_{a} = 0$ 即可保证它在 (147)、(152) 变换下不变。事实上，由于明显性质 $D^{0}q^{+a} = q^{+a}$ ，变换 (152) 保持作用量不变，而解析积分测度的变换,

$$\delta\left(du d\zeta^{(-4)}\right) = \left(du d\zeta^{(-4)}\right)\Lambda,$$

$$\Lambda := \partial_{m}\lambda^{m} + \partial^{--}\lambda^{++} - \partial^{-}_{\hat{\mu}}\lambda^{+\hat{\mu}}, \tag{156}$$

can be canceled by the appropriate analytic rescaling of q^{+a} ,

可以通过对 q^{+a} 做适当的解析标度变换抵消,

$$\delta q^{+a} = -\frac{1}{2}\Lambda q^{+a} \tag{157}$$

The vielbein coefficients H^{++M}, H^{++++} are unconstrained analytic superfields involving infinite numbers of the component fields which come from the harmonic expansions. Most of these fields, like in the $\mathcal{N} = 2$ SYM prepotential V^{++} , can be gauged away by the analytic parameters $\lambda^{M}, \lambda^{++}$, leaving in the Wess-Zumino gauge just the irreducible off-shell $(24 + 24)$ -component $\mathcal{N} = 2$ Weyl multiplet [53, 60, 61] :

vielbein 系数 H^{++M}, H^{++++} 是无约束的解析超场, 包含调和展开产生的无穷多分量场。和 $\mathcal{N}=2$ 杨-米尔斯理论中 prepotential V^{++} 的情况一样，这些场大部分都可以通过解析参数 λ^M, λ^{++} 做规范变换消去，在 Wess-Zumino 规范下仅剩下不可约脱壳 (24 + 24) 分量的 $\mathcal{N}=2$ Weyl 多重态 [53, 60, 61] :

$$H^{++m}(\zeta,u) = -2i\theta^+\sigma^a\overline{\theta}^+ e_a^m(x) + \left(\overline{\theta}^+\right)^2 \theta^{+\mu}\psi_{\mu i}^m(x)u^{-i}$$

$$+(\theta^+)^2\overline{\theta}^+_{\dot\mu}\overline{\psi}_i^{m\dot\mu}(x)u^{-i} + (\theta^+)^2\left(\overline{\theta}^+\right)^2 V_{ij}^m u^{-i}u^{-j},$$

$$H^{++\mu+}(\zeta,u) = (\theta^+)^2\overline{\theta}^+_{\dot\mu}A^{\mu\dot\mu}(x) + \left(\overline{\theta}^+\right)^2\theta_v^+ t^{(v\mu)}(x) + (\theta^+)^2\left(\overline{\theta}^+\right)^2\chi_i^\mu(x)u^{-i},$$

$$H^{++\dot\mu+}(\zeta_A,u) = \widetilde{H^{++\mu+}},$$

$$H^{+4}(\zeta,u) = (\theta^+)^2\left(\overline{\theta}^+\right)^2 D(x). \tag{158}$$

Here $e_a^m, \psi_{\mu i}^m, \overline{\psi}_{\dot\mu}^{mi}, V_{ij}^m, A^{\mu\dot\mu}$ are the conformal graviton, gravitini and gauge fields for the local $SU(2)$ and γ_5 transformations; all other fields are auxiliary.

此处 $e_a^m, \psi_{\mu i}^m, \overline{\psi}_{\dot\mu}^{mi}, V_{ij}^m, A^{\mu\dot\mu}$ 是共形引力子、引力微子，以及局域 $SU(2)$ 和 γ_5 变换的规范场；其余所有场都是辅助场。

Covariant Derivative $\mathcal{D}^{--}$ and Minimal Superconformal Action

协变导数 $\mathcal{D}^{--}$ 与极小超共形作用量

Now, armed with the superfield differential geometry formalism for Einstein $\mathcal{N}=2$ SG described in section " $\mathcal{N}=2$ Einstein Supergravity "From Scratch"," we can complete the geometric setup of conformal $\mathcal{N}=2$ SG.

现在，借助“从零开始构建 $\mathcal{N}=2$ 爱因斯坦超引力”一节中介绍的爱因斯坦 $\mathcal{N}=2$ 超引力的超场微分几何形式体系，我们可以完成共形 $\mathcal{N}=2$ 超引力的几何架构了。

First, we define the second harmonic derivative $\mathcal{D}$ -

首先，我们定义二阶调和导数 $\mathcal{D}$ -

$$\mathcal{D}^{--} = \partial_A^{--} + H^{--m}\partial_m^A + H^{--\hat\mu\pm}\partial_{\hat\mu}^{\mp A}. \tag{159}$$

To deduce the transformation rule of $\mathcal{D}^{--}$, we first note that under λ^{++} transformation, with the supervielbein H^{--M} in (161) still untouched, $\mathcal{D}^{--}$ transforms as

为了推导 $\mathcal{D}^{--}$ 的变换规则，我们首先注意到，在 λ^{++} 变换下，当 (161) 中的超标架 H^{--M} 保持不变时，$\mathcal{D}^{--}$ 的变换形式为

$$\hat{\delta}_{\lambda^{++}} \mathcal{D}^{--} = -(\mathcal{D}^{--}\lambda^{++})\, \partial_A^{--}. \tag{160}$$

Then the full covariant transformation law of $\mathcal{D}^{--}$ (with the proper transformations of the supervielbein coefficients) can naturally be chosen as

那么，考虑超标架系数的恰当变换后，$\mathcal{D}^{--}$ 的完整协变变换法则自然可以写为

$$\delta \mathcal{D}^{--} = -(\mathcal{D}^{--}\lambda^{++})\, \mathcal{D}^{--}. \tag{161}$$

The appropriate transformation rules of H^{--M} are then as follows

H^{--M} 的恰当变换规则如下

$$\delta H^{--M} = -(\mathcal{D}^{--}\lambda^{++})\, H^{--M} + \mathcal{D}^{--}\lambda^{M}. \tag{162}$$

To ensure covariance of the harmonic flatness condition, we are led to modify it as

为了保证调和平坦性条件的协变性，我们需要将其修正为

$$\left[\mathcal{D}^{++} - H^{+4}\mathcal{D}^{--}, \mathcal{D}^{--}\right] = D^{0}. \tag{163}$$

In principle, one can now repeat the same steps as in the case of Einstein $\mathcal{N} = 2$ SG: define the covariant spinor and vector derivatives, super torsions and curvatures, etc. This formalism could be of use, e.g., for solving the problem of constructing higher-derivative superconformal invariants and related problems. Here we will be interested only in constructing some off-shell actions of compensators in the background of $\mathcal{N} = 2$ Weyl multiplet. Using them, one can derive the actions of diverse versions of Einstein $\mathcal{N} = 2$SG including the one we have discussed in the previous section.

原则上，现在我们可以沿用爱因斯坦 $\mathcal{N} = 2$ 超引力情形下的相同步骤: 定义协变旋量导数、矢量导数、超挠率、超曲率等等。这套形式体系可用于解决高阶导数超共形不变量构造等相关问题。本文我们仅关注在 $\mathcal{N} = 2$ 外尔多重态背景下构造一些补偿子的离壳作用量。利用这些构造，我们可以推导出不同版本的爱因斯坦 $\mathcal{N} = 2$SG 作用量，包括我们在上一节已经讨论过的版本。

It turns out that it is easy to generalize the action (120) to the superconformal case. The relevant density E defining the invariant integration measure is still given by the same formula (119) but the objects entering it have different transformation properties as compared to those inherent to Einstein SG case, Eqs. (116), because of the presence of extra analytic gauge parameter λ^{++} . It is straightforward to find the new terms in the transformations of $e^{m}_{[\hat{\mu}\hat{\nu}]}$ and $e^{\hat{\nu}}_{\hat{\mu}}$

事实证明，将作用量 (120) 推广到超共形情形十分容易。定义不变积分测度的相关密度 E 仍由相同的公式 (119) 给出，但由于额外解析规范参数 λ^{++} 的存在，其中的对象与爱因斯坦超引力情形 (即式 (116)) 相比，具有不同的变换性质。我们可以直接得到 $e^m_{[\hat{\mu}\hat{\nu}]}$ 和 $e^{\hat{\nu}}_{\hat{\mu}}$ 变换中的新项

$$\delta_{\lambda^{++}} e^m_{[\hat{\mu}\hat{\nu}]} = -\left(\mathcal{D}^{--}\lambda^{++}\right) e^m_{[\hat{\mu}\hat{\nu}]}$$

$$-e^n_{[\hat{\mu}\hat{\upsilon}]}\left[\partial_n \lambda^{++}\left(H^{--m} - H^{--\hat{\mu}+} e^{-1\hat{\upsilon}}_{\hat{\mu}} \partial^+_{\hat{\upsilon}} H^{--m}\right)\right], \tag{164}$$

$$\delta_{\lambda^{++}} e^{\hat{\upsilon}}_{\hat{\mu}} = -\left(\mathcal{D}^{++}\lambda^{++}\right) e^{\hat{\upsilon}}_{\hat{\mu}}$$

$$-\left(\partial^+_{\hat{\mu}} H^{--m}\partial_m \lambda^{++} + e^{\hat{\rho}}_{\hat{\mu}} \partial^-_{\hat{\rho}} \lambda^{++}\right) H^{--\hat{\upsilon}+}. \tag{165}$$

The second piece in the transformations (162) has the same form as in Einstein $\mathcal{N} = 2$SG , so it produces the same weight transformation of E as in Eq. (118). So we are led to consider the impact of the transformations (164), (165) only on various determinant factors in (119). It is easy to check that the object $f^{\mu\dot{\mu}} = e^m e^{\mu\dot{\mu}}_m$ remains invariant under the second, rotational part of the transformation (164), so the square root in the definition (119) is also invariant. Then it is easy to show that

变换 (162) 中的第二部分与爱因斯坦 $\mathcal{N} = 2$SG 情形形式相同，因此它给出的 E 权变换与式 (118) 一致。因此我们仅需要分析变换 (164)、(165) 对 (119) 中各个行列式因子的影响。容易验证，$f^{\mu\dot{\mu}} = e^m e^{\mu\dot{\mu}}_m$ 在变换 (164) 的第二部分即转动部分下保持不变，因此 (119) 定义中的平方根也是不变的，进而可以证明

$$\delta_{\lambda^{++}} E = \left(\partial^{--}_A \lambda^{++} - \mathcal{D}^{--}\lambda^{++}\right) E. \tag{166}$$

Hence, the total transformation of E reads

因此，E 的总变换为

$$\delta_\lambda E = \left(\partial_m \lambda^m - \partial^+_{\hat{\upsilon}} \lambda^{-\hat{\upsilon}} - \partial^-_{\hat{\upsilon}} \lambda^{+\hat{\upsilon}} + \partial^{--}_A \lambda^{++} - \mathcal{D}^{--}\lambda^{++}\right) E. \tag{167}$$

One should also take into account that the transformation of the integration element $dud^{12}Z$ gets now a contribution from the variation of the harmonics u^+_i , namely, $\frac{\partial \lambda^{++} u^-_i}{\partial u^+_i} = \partial^{--}_A \lambda^{++}$, so that

我们还需要注意，积分单元 $dud^{12}Z$ 的变换现在会得到来自调和变量 u^+_i 变分的贡献，即 $\frac{\partial \lambda^{++} u^-_i}{\partial u^+_i} = \partial^{--}_A \lambda^{++}$ ，因此

$$\delta\left(dud^{12}Z\right) = dud^{12}Z\left(\partial_m \lambda^m + \partial^{--}_A \lambda^{++} - \partial^+_{\hat{\upsilon}} \lambda^{-\hat{\upsilon}} - \partial^-_{\hat{\upsilon}} \lambda^{+\hat{\upsilon}}\right). \tag{168}$$

Finally, we obtain

最终，我们得到

$$\delta\left(dud^{12}ZE^{-1}\right) = \left(dud^{12}ZE^{-1}\right)(\mathcal{D}^{--}\lambda^{++}). \tag{169}$$

To establish a contact with the previous section, we need to extend the set of the harmonic superspace coordinates by x^5 and, correspondingly, extend the harmonic covariant derivatives $\mathcal{D}^{\pm\pm}$ by the extra vielbeins $H^{\pm\pm 5}$,

为了和上一节建立联系，我们需要通过 x^5 扩展调和超空间坐标集，并相应地通过额外标架 $H^{\pm\pm 5}$ 扩展调和协变导数 $\mathcal{D}^{\pm\pm}$ ，

$$\mathcal{D}^{++} \Rightarrow \mathcal{D}^{++} + H^{++5}\partial_5,\ \mathcal{D}^{--} \Rightarrow \mathcal{D}^{--} + H^{--5}\partial_5. \tag{170}$$

As before, nothing depends on x^5, H^{++5} is analytic, $H^{++5} = H^{++5}(\zeta, u)$, while H^{--5} is related to H^{++5} by the proper harmonic flatness equation. In accord with the transformation properties (152),(161), $H^{\pm\pm 5}$ transform as

和之前一样，没有量依赖于 x^5, H^{++5} ， x^5, H^{++5} 是解析的，即 $H^{++5} = H^{++5}(\zeta, u)$ ， 而 H^{--5} 通过合适的调和平坦方程与 H^{++5} 关联。根据变换性质 (152)、(161)， $H^{\pm\pm 5}$ 的变换形式为

$$\delta H^{++5} = \mathcal{D}^{++}\lambda^5,\ \delta H^{--5} = -(\mathcal{D}^{--}\lambda^{++})H^{--5} + \mathcal{D}^{--}\lambda^5. \tag{171}$$

Thus we observe that the same action

因此我们可以看到，同一个作用量

$$S_{SG}^{\mathcal{N}=2} = -\frac{1}{\kappa^2}\int dud^4x_A d^4\theta_A^+ d^4\theta_A^- E^{-1}H^{++5}H^{--5} \tag{172}$$

is invariant under the full superconformal group (147) and λ_5 transformations,

在整个超共形群 (147) 和 λ_5 变换下是不变的，

$$\delta_\lambda S_{SG}^{\mathcal{N}} = 0,\ \delta_{\lambda^5} S_{SG}^{\mathcal{N}} = 2 = 0. \tag{173}$$

From Conformal to Einstein $\mathcal{N} = 2$SG

从共形到爱因斯坦 $\mathcal{N} = 2$SG

At this stage we have what is called ”minimal off-shell representation” [59,62]. It involves $(32+32)$ off-shell degrees of freedom: $(24+24)$ from Weyl $\mathcal{N} = 2$ multiplet and $(8+8)$ from the Maxwell multiplet described by H^{++5} . The scalar fields in H^{++5} can serve as compensators for the scale and R -symmetry present in superconformal gauge group. However, the $\mathcal{N} = 2$ superconformal action (172), though being formally identical to that of $\mathcal{N} = 2$ Einstein SG, Eq. (120), involves a wider set of fields and a wider set of invariances. As it stands, it is in fact inconsistent and therefore cannot serve as the appropriate action neither for $\mathcal{N}=2$ Einstein SG nor for $\mathcal{N}=2$ superconformal SG. The reason was explained in detail in [14]. In brief, the component action in WZ gauges for Weyl and Maxwell supermultiplets contains an unwanted term

至此我们得到了所谓的“最小离壳表示”[59,62]。它包含 $(32+32)$ 个离壳自由度: $(24+24)$ 个来自外尔 $\mathcal{N}=2$ 多重态，$(8+8)$ 个来自 H^{++5} 描述的麦克斯韦多重态。H^{++5} 中的标量场可作为超共形规范群中标度和 R 对称性的补偿器。然而，$\mathcal{N}=2$ 超共形作用量 (172) 虽然形式上与 $\mathcal{N}=2$ 爱因斯坦超引力的式 (120) 相同，但包含更广泛的场集合与更广泛的不变性。就目前形式而言，它实际上不自洽，因此既不能作为 $\mathcal{N}=2$ 爱因斯坦超引力也不能作为 $\mathcal{N}=2$ 超共形超引力的合适作用量。原因已在文献 [14] 中详细说明。简而言之，外尔与麦克斯韦超多重态在 WZ 规范下的分量作用量包含一个多余项

$$\sim\int d^4xD\varphi^2 \tag{174}$$

where D is the auxiliary field from H^{+4} (see Eq. (158)) and φ is the imaginary part of the complex scalar field of H^{++5} . Then varying D in (174) yields meaningless constraint $\varphi^2=0$ (The non-existence of consistent invariant action for the ”minimal representation” in the component approach was shown in [62].). To evade this trouble, one needs one more compensating superfield besides H^{++5} , so as to compensate the local $SU(2)$ with the analytic parameter $\lambda^{++}(\zeta,u)$.

其中 D 是来自 H^{+4} 的辅助场 (见式 (158))，φ 是 H^{++5} 复标量场的虚部。对 (174) 中的 D 变分会得到无意义的约束 $\varphi^2=0$ (分量方法中“最小表示”不存在自洽不变作用量已在文献 [62] 中证明。)。为规避这一问题，除 H^{++5} 外还需要一个额外的补偿超场，以用解析参数 $\lambda^{++}(\zeta,u)$ 补偿定域 $SU(2)$ 。

The simplest variant is to consider an extended system involving the so-called nonlinear supermultiplet [59]. In the flat HSS it is described by the analytic superfield N^{++} obeying the constraint [14]

最简单的方案是考虑包含所谓非线性超多重态的推广系统 [59]。在平坦调和超空间中，它由满足约束条件 [14] 的解析超场 N^{++} 描述

$$D^{++}N^{++}+\left(N^{++}\right)^2=0. \tag{175}$$

It contains $8+8$ independent off-shell components. The curved covariant generalization of (175) is as follows

它包含 $8+8$ 个独立的离壳分量。(175) 的弯曲协变推广如下

$$\mathcal{D}^{++}N^{++}+\left(N^{++}\right)^2-H^{+4}=0, \tag{176}$$

provided that N^{++} has the following transformation law

前提是 N^{++} 满足如下变换规律

$$\delta N^{++}=\lambda^{++}. \tag{177}$$

One observes that the covariantized constraint (176) can serve as the definition of H^{+4}

可以看到，协变化约束 (176) 可以作为 H^{+4} 的定义

$$H^{+4} = \mathcal{D}^{++}N^{++} + (N^{++})^2. \tag{178}$$

Furthermore, the transformation law (177) tells us that one can make use of the super parameter $\lambda^{++}(\zeta, u)$ to entirely gauge away N^{++}

此外，变换规律 (177) 表明，我们可以利用超参数 $\lambda^{++}(\zeta, u)$ 完全规范固定掉 N^{++}

$$N^{++} = 0 \Rightarrow H^{+4} = 0. \tag{179}$$

In this gauge, the superconformal diffeomorphisms (147) are reduced to (83), the covariant derivatives $\mathcal{D}^{\pm\pm}$ to those defined in the previous section and the action (172) to the previously given $\mathcal{N} = 2$ SG action (120). Of course, the possibility to choose the gauge (179) in (120) is ensured by the superconformal invariance of this action. The trouble mentioned earlier is automatically resolved, as $D = 0$ in the gauge (179). Note that no superconformally invariant action of N^{++} alone can be constructed even in the flat case. The same is true also in the curved case, and so the version of $\mathcal{N} = 2$ Einstein SG with N^{++} as a compensator is described by the superconformal Maxwell action (172), with the additional superconformally covariant constraint (176) imposed ”by hand”.

在该规范下，超共形微分同胚 (147) 约化为 (83)，协变导数 $\mathcal{D}^{\pm\pm}$ 约化为上一节定义的形式，作用量 (172) 约化为之前给出的 $\mathcal{N} = 2$ 超引力作用量 (120)。当然，作用量 (120) 可以选择规范 (179) 由该作用量的超共形不变性保证。前文提到的问题会自动解决，因为在规范 (179) 中 $D = 0$ 。注意，即使在平坦情形下，也无法构造仅包含 N^{++} 的超共形不变作用量。弯曲情形同样如此，因此以 N^{++} 为补偿器的 $\mathcal{N} = 2$ 爱因斯坦超引力版本由超共形麦克斯韦作用量 (172) 描述，并“手动”附加了超共形协变约束 (176)。

It is possible to choose as second compensator other $\mathcal{N} = 2$ matter supermulti-plets with finite numbers of the off-shell components, for which superconformally invariant actions exist in the flat HSS. Such are the so called ”improved” tensor $\mathcal{N} = 2$ multiplet with the off-shell content $8 + 8$ [63] and the hypermultiplet with the non-trivially realized operator central charge [28]. The HSS formulation of these off-shell multiplets were given in [14] and [64,65]. All these multiplets, like N^{++} , are described by analytic superfields subjected to some covariant differential constraints. The superfield actions of the relevant versions of Einstein $\mathcal{N} = 2$ SG are sums of the superconformal Maxwell action (172) and the actions of the compensating matter superfields in the background of $\mathcal{N} = 2$ Weyl multiplet. The aforementioned trouble with the auxiliary field D is solved in all cases due to the property that the field D serves as a Lagrange multiplier identifying some scalar fields from the matter compensator with the scalar field φ from the Maxwell compensator action. In the appropriate gauge the remaining scalar field ϕ plays the role of dilaton and the relevant part of the gravity sector of the full action takes the form

我们也可以选择其他具有有限离壳分量的 $\mathcal{N}=2$ 物质超多重态作为第二个补偿器，这类超多重态在平坦调和超空间中存在超共形不变作用量。这类就是所谓的“改进”张量 $\mathcal{N}=2$ 多重态，离壳内容为 $8+8$ [63]，以及具有非平凡实现算符中心荷的超多重态 [28]。这些离壳多重态的调和超空间表述已在文献 [14] 和 [64,65] 中给出。所有这些多重态都和 N^{++} 一样，由满足某些协变微分约束的解析超场描述。对应形式的爱因斯坦 $\mathcal{N}=2$ 超引力的超场作用量，是超共形麦克斯韦作用量 (172) 与 $\mathcal{N}=2$ 外尔多重态背景下补偿物质超场作用量的和。上述辅助场 D 的问题在所有情况下都能解决，因为场 D 作为拉格朗日乘子，将物质补偿器中的某些标量场与麦克斯韦补偿器作用量中的标量场 φ 对应起来。在合适的规范下，剩余标量场 ϕ 扮演 dilation 的角色，完整作用量引力部分的对应形式为

$$S_{\text{grav}}=\int d^4x\sqrt{-g}\Big[3\phi\Big(\Box-\frac{1}{6}R\Big)\phi+\xi^2\phi^4\Big], \tag{180}$$

where $\Box=\nabla^m\partial_m$ and the second term corresponds to the cosmological constant. This action is invariant under local rescalings

其中 $\Box=\nabla^m\partial_m$ ，第二项对应宇宙学常数。该作用量在局域标度变换下不变

$$\delta g_{mn}(x)=2a(x)\,g_{mn}(x)\,,\delta\phi(x)=-a(x)\,\phi(x)\,. \tag{181}$$

Fixing this extra gauge invariance as $\phi(x)=\kappa^{-1}$, we reproduce the standard Einstein action

如 $\phi(x)=\kappa^{-1}$ 固定这个额外规范不变性后，我们就可以得到标准的爱因斯坦作用量

$$S_{\text{Ein}}=\int d^4x\sqrt{-g}\Big(-\frac{1}{2\kappa^2}R+\frac{\xi^2}{\kappa^4}\Big). \tag{182}$$

This mechanism [66] is the $\mathcal{N}=0$ prototype of the general compensating procedure and it works in all known off-shell versions of $\mathcal{N}=2$ supergravity.

这个机制 [66] 是一般补偿过程的 $\mathcal{N}=0$ 原型，它适用于所有已知的 $\mathcal{N}=2$ 超引力离壳版本。

The versions of $\mathcal{N}=2$ Einstein SG with the 8+8 compensators mentioned above exhibit the sets of $40+40$ off-shell components in WZ gauge, like in the minimal version with nonlinear multiplet N^{++} as a compensator. The locally superconformal action of the improved tensor multiplet in HSS [67] is given by

上述带有 8+8 个补偿器的 $\mathcal{N}=2$ 爱因斯坦超引力版本，在 WZ 规范中共有 40+40 个离壳分量，和以非线性多重态 N^{++} 为补偿器的最小版本情况一致。调和超空间中改进张量多重态的局域超共形作用量 [67] 由下式给出

$$S_{\text{impr}}\sim\int dud\zeta^{(-4)}\left[\left(g^{++}\right)^2-\Gamma^{++}g^{++}c^{+-}-H^{+4}\left(1+2g^{++}c^{--}\right)\right], \tag{183}$$

where

其中

$$g^{++} = \frac{\ell^{++}}{1+\sqrt{1+\ell^{++}c^{--}}}, \ \ell^{++} = L^{++} - c^{++}, c^{\pm\pm} = c^{ik}u_i^{\pm}u_k^{\pm} \tag{184}$$

and the following covariant off-shell constraint and the transformation properties hold

且满足如下协变离壳约束和变换性质

$$(\mathcal{D}^{++} + \Gamma^{++})L^{++} = 0, \ \Gamma^{++} := (-1)^{P(M)}\partial_M H^{++M}, \tag{185}$$

$$\delta\Gamma^{++} = 2\lambda^{++} + \mathcal{D}^{++}\Lambda, \ \delta L^{++} = -\Lambda L^{++}, \tag{186}$$

where Λ was defined in (156). The superfield L^{++} involves $8+8$ off-shell components, which, together with $32+32$ components of the "minimal off-shell representation," yield total of $40+40$ off-shell components.

其中 Λ 已在式 (156) 中定义。超场 L^{++} 包含 $8+8$ 个离壳分量，加上“最小离壳表示”的 32+32 个分量，总共有 40+40 个离壳分量。

Analogously, the compensating hypermultiplet of Ref. [28] is described by the x^5 -dependent analytic superfield $\phi^{+a}\left(\zeta, x^5, u\right)$, which is subjected to the off-shell constraint

类似地，文献 [28] 中的补偿超多重态由依赖 x^5 的解析超场 $\phi^{+a}\left(\zeta, x^5, u\right)$ 描述，该超场满足离壳约束

$$\left(\mathcal{D}^{++} + H^{++5}\partial_5 + \frac{1}{2}\Gamma^{++}\right)\phi^{+a} = 0, \tag{187}$$

and possesses the transformation rules

且具有如下变换规则

$$\delta_\lambda \phi^{+a} = -\frac{1}{2}\Lambda\phi^{+a}, \ \delta_{\lambda^5}\phi^{+a} = -\lambda^5\partial_5\phi^{+a}. \tag{188}$$

Its HSS action is written as [14]

它的调和超空间作用量写为 [14]

$$S_{\phi^+} = -\frac{1}{2}\int dud\zeta^{(-4)}H^{++5}\phi^{+a}\partial_5\phi^+_a. \tag{189}$$

As distinct from the q^+ superfield with an infinite number of off-shell component fields, ϕ^{+a} amounts to the finite off-shell set of $8+8$ components. It is easy to check that the constraint (187) and the action (189) are superconformally invariant. Their λ^5 invariance can be checked using the constraint (187) itself and the transformation law $\delta_{\lambda^5}H^{++5} = \mathcal{D}^{++}\lambda^5$. Also, it is straightforward to show that $\partial_5\left[H^{++5}\phi^{+a}\partial_5\phi^+_a\right]$ is a total derivative, so there is no need to integrate over x^5 in (189). Indeed, using (187) and the identity $\partial_5\phi^a\partial_5\phi_a = 0$, one obtains

和含有无穷多离壳分量场的 q^+ 超场不同，ϕ^{+a} 只含有有限个离壳分量，共 $8+8$ 个。不难验证约束 (187) 和作用量 (189) 是超共形不变的。它们的 λ^5 不变性可以利用约束 (187) 本身和变换律 $\delta_{\lambda^5} H^{++5} = \mathcal{D}^{++}\lambda^5$ 验证。此外，我们可以直接证明 $\partial_5\left[H^{++5}\phi^{+a}\partial_5\phi^+_a\right]$ 是全导数，因此不需要在 (189) 中对 x^5 积分。实际上，利用 (187) 和恒等式 $\partial_5\phi^a\partial_5\phi_a = 0$，我们可以得到

$$\partial_5\left[H^{++5}\phi^{+a}\partial_5\phi^+_a\right] \simeq \left[\left(\mathcal{D}^{++} + \frac{1}{2}\Gamma^{++}\right)\phi^{+a}\right]\partial_5\phi_a,$$

where $\simeq$ means "up to a total derivative." Applying once more the constraint (187) and making use of the identity $\partial_5\phi^a\partial_5\phi_a = 0$, we are convinced that this expression is vanishing.

其中 $\simeq$ 表示“差一个全导数”。再次应用约束条件 (187) 并利用恒等式 $\partial_5\phi^a\partial_5\phi_a = 0$，可证该表达式为零。

Principal Version of Einstein $\mathcal{N} = 2$ Supergravity and General Matter Couplings

爱因斯坦 $\mathcal{N} = 2$ 超引力的主形式与一般物质耦合

An important problem relevant to any kind of supergravity is how to construct its most general couplings to matter which would extend those known in the rigid supersymmetry. As was mentioned earlier, the most general $\mathcal{N} = 2$ matter in the rigid case is described by sigma models with general HK bosonic target spaces [37]. As was shown in [68], the bosonic target manifolds of $\mathcal{N} = 2$ sigma models in the Einstein supergravity background are quaternion-Kähler (QK) [68], in contrast to the HK ones in the flat $\mathcal{N} = 2$ case (QK manifolds are $4n$ -dimensional Riemannian manifolds with the holonomy group in $Sp(n) \times Sp(1)$. In the QK sigma models in the $\mathcal{N} = 2$ SG background the $Sp(1)$ curvature R can be normalized so that $R = -8n(n+2)\kappa^2$.). The latter are described by the analytic superspace action (60) involving $2n$ unconstrained hypermultiplet superfields $q^{+a}(\zeta), a = 1, \ldots, 2n$, the bosonic components of which parametrize $4n$ -dimensional HK manifolds, with an arbitrary interaction Lagrangian $L^{+4}(q^+, u)$ called the HK potential. Any HK manifold corresponds to a definite HK potential, and vice versa, any L^{+4} after passing to component fields generates the target bosonic metric, which is guaranteed to be HK [38,39]. So the $\mathcal{N} = 2$ hypermultiplet actions provides a powerful method of explicit calculation of diverse HK metric, and as such it was used, e.g., in [39-41].

任何超引力都存在一个相关的重要问题: 如何构建它与物质的最一般耦合，从而推广刚性超对称中已知的耦合形式。如前文所述，刚性情形下最一般的 $\mathcal{N} = 2$ 物质由带有一般 HK 玻色目标空间的 sigma 模型描述 [37]。文献 [68] 已证明，爱因斯坦超引力背景下 $\mathcal{N} = 2$ sigma 模型的玻色目标流形是四元数-凯勒 (QK) 流形 [68]，这与平坦 $\mathcal{N} = 2$ 情形下的 HK 流形形成对比 (QK 流形是全曲率群包含于 $Sp(n) \times Sp(1)$ 的 $4n$ 维黎曼流形。在 $\mathcal{N} = 2$ 超引力背景下的 QK sigma 模型中，$Sp(1)$ 曲率 R 可被归一化为满足 $R = -8n(n+2)\kappa^2$ 。)。刚性 HK 超多重子由解析超空间作用量 (60) 描述，该作用量包含 $2n$ 个无约束超多重子超场 $q^{+a}(\zeta), a = 1, \ldots, 2n$，其玻色分量参数化 $4n$ 维 HK 流形，带有任意相互作用拉格朗日量 $L^{+4}(q^+, u)$，称为 HK 势。任何 HK 流形都对应一个确定的 HK 势，反之，任何 L^{+4} 在转为分量场后都会生成目标玻色度量，且该度量必然是 HK 度量 [38,39]。因此 $\mathcal{N} = 2$ 超多重子作用量是计算各类 HK 度量的有力方法，例如文献 [39-41] 就采用了这一方法。

The problem of generalization of these hypermultiplet couplings to the case of $\mathcal{N} = 2$ SG required, first of all, finding out the appropriate density ensuring the invariance of the analytic HSS integration measure. Indeed, the relevant actions were expected to generalize the rigid $\mathcal{N} = 2$ supersymmetric ones, which are defined just as integrals over the analytic HSS. Unfortunately, there is no way to construct such densities in the framework of the standard $\mathcal{N} = 2$ SG versions based on the compensators with finite sets of off-shell fields. For instance, using the constrained hypermultiplet with central charge $\phi^a\left(\zeta, x^5, u\right)$), one can construct the analytic density $\left(u^{+a}\phi_a\right)^2$ with he transformation properties required, $\delta_\lambda\left(u^{+a}\phi_a\right)^2 = -\Lambda\left(u^{+a}\phi_a\right)^2$, but it depends on x^5 and so cannot be used for constructing invariant $4D$ actions. In the case of improved tensor multiplet as a compensator, it is also impossible to construct the compensating density from the basic objects of the relevant action, $\left(\ell^{++}(\zeta, u), c^{(ik)}\right)$.

要将这些超多重子耦合推广到 $\mathcal{N} = 2$ 超引力情形，首先需要找到合适的密度以保证解析 HSS(超空间) 积分测度的不变性。的确，人们曾预期相关作用量会推广刚性 $\mathcal{N} = 2$ 超对称作用量——后者本身就是定义在解析 HSS 上的积分。遗憾的是，在基于带有限脱壳场补偿子的标准 $\mathcal{N} = 2$ 超引力形式框架内，无法构造出这类密度。例如，使用带中心荷 $\phi^a\left(\zeta, x^5, u\right)$ 的约束超多重子，可以构造出满足要求变换性质的解析密度 $\left(u^{+a}\phi_a\right)^2$ ，即 $\delta_\lambda\left(u^{+a}\phi_a\right)^2 = -\Lambda\left(u^{+a}\phi_a\right)^2$ ，但它依赖于 x^5 ，因此无法用于构造不变的 $4D$ 作用量。当改进张量多重子作为补偿子时，也无法从相关作用量的基本对象构造出补偿密度，即 $\left(\ell^{++}(\zeta, u), c^{(ik)}\right)$ 。

An exception is the so called "principal" version of $\mathcal{N} = 2$SG , which is based on the choice, as a compensator, of the unconstrained q^+ hypermultiplet with infinitely many auxiliary fields off shell. It is a completely novel possibility suggested by the HSS approach. It could not be discovered in the approaches using the component fields or constrained ordinary $\mathcal{N} = 2$ superfields.

一个例外是所谓的 $\mathcal{N} = 2$SG “主”形式，它选择带有无穷多脱壳辅助场的无约束 q^+ 超多重子作为补偿子。这是 HSS 方法提出的一种全新可能性，在使用分量场或约束常规 $\mathcal{N} = 2$ 超场的研究方法中无法被发现。

This version can be derived in the following way, starting from the version with nonlinear multiplet N^{++} as a compensator. Insert the constraint (175) into the action with the proper superfield Lagrange multiplier

该版本可通过下述方法推导得到: 从以非线性多重态 N^{++} 作为补偿器的版本出发，将约束 (175) 代入带有恰当超场拉格朗日乘子的作用量中

$$S_{\omega,N} = \frac{1}{2}\int du d\zeta^{(-4)}\omega^2\left[H^{+4} - \mathcal{D}^{++}N^{++} - \left(N^{++}\right)^2\right]. \tag{190}$$

To secure superconformal invariance, one is led to ascribe to ω the transformation property

为保证超共形不变性，我们需要赋予 ω 如下变换性质

$$\delta\omega = -\frac{1}{2}\Lambda\omega. \tag{191}$$

Now we observe that the action (190) is none other than a change of variables in the covariantized q^+ action

现在我们注意到，作用量 (190) 无非就是协变 q^+ 作用量中的一次变量替换

$$S_{\omega,N} = -\frac{1}{2}\int dud\zeta^{(-4)} q_i^+ \mathcal{D}^{++} q^{++},\ q_i^+ = \left(u_i^+ - N^{++} u_i^-\right)\omega,$$

$$\omega = u_i^- q^{+i},\ N^{++} = \frac{u_i^+ q^{+i}}{u_j^- q^{+j}},\ \delta q_i^+ = -\frac{1}{2}\Lambda q_i^+. \tag{192}$$

So the total action of this version of Einstein $\mathcal{N}=2$ SG can be equivalently written as

因此该版本爱因斯坦 $\mathcal{N}=2$ 超引力的总作用量可以等价地写为

$$S_{N=2}^{\text{prin}} = -\frac{1}{\kappa^2}\int dud^4x_A d^4\theta_A^+ d^4\theta_A^- E^{-1} H^{++5} H^{--5}$$

$$-\frac{1}{2}\int dud\zeta^{(-4)}\left[q_i^+ \widehat{\mathcal{D}}^{++} q^{+i} - i\frac{\xi}{\kappa}(\tau_3)^{ik} q_i^+ H^{++5} q_k^+\right], \tag{193}$$

where we denoted $\widehat{\mathcal{D}}^{++}$ the part of $\mathcal{D}^{++}$ independent of H^{++5} and added the term generating in components the cosmological constant $\sim \xi$. This term appears after identifying, a lá Scherk and Schwarz, the derivative ∂_5 with the $U(1)$ generator of $SU(2)$ acting on the doublet indices and commuting with supersymmetry, $\partial_5 q_j^+ = i\frac{\xi}{\kappa}(\tau_3)_j^k \lambda^5 q_k^+$. Note that the Maxwell and hypermultiplet parts of the action have wrong signs, which matches with the role of the relevant multiplets as the compensating ones.

其中我们用 $\widehat{\mathcal{D}}^{++}$ 表示 $\mathcal{D}^{++}$ 中不依赖于 H^{++5} 的部分，并额外添加了分量形式生成宇宙学常数 $\sim \xi$ 的项。按照谢尔克和施瓦茨的方法，将导数 ∂_5 等同为作用于双态指标、与超对易的 $U(1)$ 生成元 $SU(2)$ 后，就得到了该项 $\partial_5 q_j^+ = i\frac{\xi}{\kappa}(\tau_3)_j^k \lambda^5 q_k^+$ 。需要注意的是，作用量中麦克斯韦部分和超多重态部分的符号是错误的，这与这些多重态作为补偿多重态的作用一致。

The harmonic superspace approach clearly exhibits the property that the bosonic target space of sigma models coupled to $\mathcal{N}=2$SG is QK [18,69]. Moreover, it offers an efficient tool of the explicit calculation of QK metrics [69, 70]. It was shown in [70] that the most general Lagrangian of hypermultiplets in the background of conformal $\mathcal{N}=2$SG , one of these hypermultiplet just being a compensator, provides the QK superfield potential which encodes any bosonic QK metric. It passes into the general HK potential $L^{+4}(q^+, u)$ when the Newton-Einstein coupling constant κ trends to zero. So it is just the principal version of $\mathcal{N}=2$ Einstein supergravity which admits the most general matter couplings and so gives rise to the generic QK sigma models in its bosonic sector, in a complete agreement with the theorem of Bagger and Witten [68].

调和超空间方法清晰地展现出以下性质: 耦合到 $\mathcal{N}=2$SG 的 sigma 模型的玻色目标空间是 QK 空间 [18,69]。此外，该方法还为 QK 度规的显式计算提供了有效工具 [69, 70]。文献 [70] 表明，共形 $\mathcal{N}=2$SG 背景下超多重态的最一般拉格朗日量 (其中一个超多重态恰好就是补偿器) 给出了 QK 超场势，该势可以编码任意玻色 QK 度规。当牛顿-爱因斯坦耦合常数 κ 趋近于零时，该势就过渡到一般 HK 势 $L^{+4}(q^+, u)$ 。因此，正是主版本的 $\mathcal{N}=2$ 爱因斯坦超引力允许最一般的物质耦合，从而在其玻色子部分得到一般 QK sigma 模型，这与巴格和威滕的定理 [68] 完全一致。

The crucial attractive feature of the principal version is the existence of the analytic density $\omega = u_i^- q^{+i}$ compensating the gauge transformation of the analytic superspace integration measure. No such density can be defined in other versions, associated with the matter compensating superfields possessing finite numbers of auxiliary fields and subjected to some constraints. Just due to this remarkable property, one can couple to conformal $\mathcal{N}=2$SG an arbitrary $\mathcal{N}=2$ sigma model. One introduces $2n$ hypermultiplets $Q^{+A}, A=1,\ldots,2n$ having zero weight with respect to $\mathcal{N}=2$ superconformal group and constructs the following locally $\mathcal{N}=2$ superconformal action directly generalizing the rigidly supersymmetric action (60)

主版本一个极为吸引人的核心性质是，它存在解析密度 $\omega=u^-_i q^{+i}$ ，可以补偿解析超空间积分测度的规范变换。在其他版本中，这类密度无法定义，因为其他版本关联的物质补偿超场具有有限个辅助场，且受一定约束限制。正是得益于这个出色的性质，我们才能将任意 $\mathcal{N}=2$ sigma 模型耦合到共形 $\mathcal{N}=2$SG 。我们引入相对于 $\mathcal{N}=2$ 超共形群权重为零的 $2n$ 超多重态 $Q^{+A}, A=1,\ldots,2n$ ，并构造如下局域 $\mathcal{N}=2$ 超共形作用量，该作用量直接推广了刚性超对称作用量 (60)

$$S_{q,Q}=\frac{1}{2}\int du d\zeta^{(-4)}\left\{-q^+_i\mathcal{D}^{++}q^{+i}+\left(u^-_i q^{+i}\right)^2\left[Q^+_A\mathcal{D}^{++}Q^{+A}\right.\right.$$

$$\left.\left.+\mathcal{L}^{+4}\left(Q^+,v^+,u^-\right)\right]\right\}, \tag{194}$$

$$v^{+i} := \frac{q^{+i}}{u^-_i q^{+i}}$$

(for simplicity, we chose $\xi=0$). The target bosonic geometry associated with this action is QK as distinct from the HK geometry associated with the action (60). The QK potential $\mathcal{L}^{+4}$, like the HK one, L^{+4} , is an arbitrary function of its arguments and it encodes the whole set of QK metrics including those without isometries. On the other hand, while coupling conformal SG to the compensators with finite sets of off-shell components, the matter-SG couplings prove to be inevitably restricted just because such compensators are subject to the proper constraints. For instance, the matter hypermultiplets self-couplings in this case must reveal some isometries. More details on the HSS description of QK sigma models and specific examples can be found in [70]. In particular, the general algorithmic scheme of deriving the target QK metrics from the action (194) was presented there. Also, another convenient trick essentially simplifying the derivation of QK metrics with isometries, the superfield quotient construction, was worked out.

(为简化起见，我们选择 $\xi=0$)。该作用量对应的目标玻色几何是四元数凯勒 (QK) 几何，与作用量 (60) 对应的哈密顿凯勒 (HK) 几何不同。QK 势 $\mathcal{L}^{+4}$ 和 HK 势一样，即 L^{+4} ，是其自变量的任意函数，它编码了整套 QK 度规，包括没有等距的 QK 度规。另一方面，当共形超引力 (SG) 与有限离 shell 分量的补偿场耦合时，物质-SG 耦合会不可避免地受到限制，原因正是这类补偿场满足特定约束。例如，该情形下物质超多重态的自耦合必须存在某些等距。关于 QK sigma 模型的 HSS 描述和具体实例，更多细节可参见文献 [70]。该文献尤其给出了从作用量 (194) 推导目标 QK 度规的通用算法框架。此外，文献还发展出一种能大幅简化带等距 QK 度规推导的便捷技巧——超场商构造。

Note that the central charge ∂_5 , the inclusion of which is necessary for gaining a non-zero cosmological constant ξ , nontrivially acts on the compensator q^{+i} in (194) (see (193)), and the invariance of (194) under ∂_5 can be achieved only for some special choices of the QK potential $\mathcal{L}^{+4}$. In other words, in the presence of cosmological constant the class of admissible target QK metrics gets essentially restricted.

需要注意，中心荷 ∂_5 (要得到非零宇宙学常数 ξ 必须引入它) 会对 (194) 中的补偿场 q^{+i} 产生非平凡作用 (见 (193))，只有对 QK 势 $\mathcal{L}^{+4}$ 做某些特殊选取，才能保证 (194) 在 ∂_5 变换下不变。换句话说，存在宇宙学常数时，允许的目标 QK 度规的类别会受到大幅限制。

It is also worth noting that in the case of pure Einstein $\mathcal{N} = 2$ SG (without matter couplings) various off-shell SG versions become equivalent via some kind of duality transformations. For instance, the action of the improved tensor multiplet (183) can be extended by adding the relevant harmonic constraint to the action with the analytic Lagrange multiplier $\omega'(\zeta)$,

还需要指出，在纯爱因斯坦 $\mathcal{N} = 2$ 超引力 (无物质耦合) 的情形下，各类离 shell 超引力版本可以通过某种对偶变换等价。例如，改进张量多重态的作用量 (183) 可以通过添加相关调和约束，延拓为带解析拉格朗日乘子 $\omega'(\zeta)$ 的作用量，

$$S_{\text{impr}} \Rightarrow S_{\text{impr}} + \int dud\zeta^{(-4)}\omega'\left(\mathcal{D}^{++} + \Gamma^{++}\right)L^{++}, \ \delta_\lambda\omega' = 0. \tag{195}$$

Using more sophisticated change of variables compared to (192), the superfields (ℓ^{++}, ω') can also be combined into the hypermultiplet $q^{+i'}$ so that the modified action (195) will transform into the covariantized action $\sim q^{+\prime}_i \mathcal{D}^{++} q^{+i'}$, thus demonstrating that the corresponding $\mathcal{N} = 2$SG is also duality-equivalent to the principal version. No such an equivalency holds after switching on couplings to matter. This is because, after integrating out the relevant Lagrange multipliers like ω or ω' , the arising constraints on the superfields like l^{++} or N^{++} will be heavily modified by the matter superfields. For instance, making the change of variables (192) in the action (194), we transform it to the expression

利用比 (192) 更复杂的变量替换，超场 (ℓ^{++}, ω') 也可以组合为超多重态 $q^{+i'}$ ，使得修正后的作用量 (195) 变换为协变化作用量 $\sim q^{+\prime}_i \mathcal{D}^{++} q^{+i'}$ ，由此证明对应的 $\mathcal{N} = 2$SG 与主版本也是对偶等价的。开启物质耦合后，这种等价性就不复存在了。这是因为，积去相关的拉格朗日乘子 (如 ω 或 ω') 后，对超场 (如 l^{++} 或 N^{++}) 得到的约束会被物质超场大幅修改。例如，对作用量 (194) 做变量替换 (192)，我们可将其变换为如下表达式

$$S^{gen}_{\omega,N} = \frac{1}{2}\int dud\zeta^{(-4)}\omega^2 \Big[H^{+4} - \mathcal{D}^{++}N^{++} - \left(N^{++}\right)^2 + Q^+_A\mathcal{D}^{++}Q^{+A}$$
$$+\mathcal{L}^{+4}\left(Q^+, u^+ - N^{++}u^-, u^-\right)\Big] \, . \tag{196}$$

It is obvious that the superfields Q^{+A} give an essential non-vanishing contribution to the modified constraint on N^{++} obtained by varying with respect to ω in (196). In the gauge $N^{++} = 0$ this constraint yields a complicated expression for the harmonic vielbein H^{+4} in terms of matter hypermultiplets.

显然，超场 Q^{+A} 会对 (196) 中对 ω 变分得到的 N^{++} 的修正约束产生重要的非零贡献。在规范 $N^{++} = 0$ 下，该约束给出了调和标架 H^{+4} 用物质超多重态表示的复杂表达式。

Summary and Some Further Problems

总结与若干后续问题

In this short overview of the HSS approach to conformal and Einstein $\mathcal{N}=2$ supergravities, we basically focused on their symmetry and geometric structures, leaving aside the issues of comparing with the component formulations and/or those in the standard and projective superspaces (see, e.g., [71, 72]). The main novel advantageous feature of the HSS formulations (compared to other ones) is that all basic off-shell $\mathcal{N}=2$ SG multiplets (Weyl multiplet and diverse compensating and matter multiplets) are accommodated by unconstrained analytic harmonic $\mathcal{N}=2$ superfields, like it takes place in rigid $\mathcal{N}=2$ supersymmetry for SYM and hypermultiplet matter multiplets. This analogy becomes especially striking for the principal version of Einstein $\mathcal{N}=2$SG , which amounts to coupling of $\mathcal{N}=2$ Weyl multiplet described by unconstrained analytic harmonic vielbeins covariantizing the harmonic derivative D^{++} , to the hypermultiplet compensator described by an unconstrained analytic superfield q^{+a} . The local superconformal $\mathcal{N}=2$ group has the fundamental realization as the harmonic analyticity-preserving diffeomorphisms of HSS. The principal version of Einstein $\mathcal{N}=2$ SG involves an infinite number of auxiliary fields and its invention is the main outcome of the HSS approach in application to the supergravity theory. It could not be discovered in any other formulation and ensures the most general $\mathcal{N}=2$ SG - matter coupling. From mathematical point of view, the corresponding superfield Lagrangian makes manifest the one-to-one correspondence between local $\mathcal{N}=2$ supersymmetry and quaternion-Kähler sigma models [68], since the basic object of this Lagrangian, $\mathcal{L}^{+4}(Q^+, v^+, u^-)$, is just the unconstrained basic object of the QK geometry, the QK potential. It is the direct analog of Kähler potential $K\left(\Phi_A, \overline{\Phi}_B\right)$ in $\mathcal{N}=1$ supersymmetry (see Eq. (13)) and HK potential in rigid $\mathcal{N}=2$ supersymmetry $L^{+4}\left(q^+, u^{\pm}\right)$ (see Eq. (60)). The general Lagrangian of the principal version of $\mathcal{N}=2$SG in interaction with matter hypermultiplet superfields given by Eq. (194) can serve as an efficient tool of explicit calculation of the QK metrics, both the already known and yet unknown ones.

在这篇关于共形与爱因斯坦 $\mathcal{N}=2$ 超引力的 HSS 方法的简短综述中，我们基本聚焦于它们的对称性与几何结构，暂不讨论与分量表述、(或) 标准与投影超空间中的表述进行比对的相关问题 (参见例如文献 [71, 72])。与其他表述相比，HSS 表述最突出的新颖优势在于: 所有基本的离壳 $\mathcal{N}=2$ 超引力 (SG) 多重态 (外尔多重态、各类补偿多重态与物质多重态) 都可由无约束解析调和 $\mathcal{N}=2$ 超场容纳，就像刚性 $\mathcal{N}=2$ 超对称中 SYM 与超多重态物质多重态的情况一样。这种类比对于爱因斯坦 $\mathcal{N}=2$SG 的主形式尤为显著，它相当于将由无约束解析调和标架描述、协变化调和导数 D^{++} 的 $\mathcal{N}=2$ 外尔多重态，与由无约束解析超场 q^{+a} 描述的超多重态补偿器耦合。局域超共形 $\mathcal{N}=2$ 群作为保持 HSS 调和解析性的微分同胚，实现了其基本表示。爱因斯坦 $\mathcal{N}=2$ 超引力 (SG) 的主形式包含无穷多个辅助场，它的提出是 HSS 方法应用于超引力理论的主要成果。它无法在其他任何表述中被发现，并且能给出最广义的 $\mathcal{N}=2$ SG-物质耦合。从数学角度看，对应的超场拉格朗日量清晰体现出局域 $\mathcal{N}=2$ 超对称与四元数-凯勒 sigma 模型之间的一一对应关系 [68]，因为该拉格朗日量的基本对象 $\mathcal{L}^{+4}(Q^+, v^+, u^-)$ 正是 QK 几何中无约束的基本对象，即 QK 势。它直接对应 $\mathcal{N}=1$ 超对称中的凯勒势 $K\left(\Phi_A, \overline{\Phi}_B\right)$ (参见式 (13)) 与刚性 $\mathcal{N}=2$ 超对称 $L^{+4}\left(q^+, u^{\pm}\right)$ 中的 HK 势 (参见式 (60))。式 (194) 给出的与物质超多重态超场相互作用的 $\mathcal{N}=2$SG 主形式的广义拉格朗日量，可作为高效工具，用于显式计算已知和未知的 QK 度量。

The HSS approach bears a close relationship to the famous twistor theory. Common for both is an extension of space-time (in twistor theory) and superspace (in the harmonic superspace approach) by a two-dimensional sphere S^2 . In such an extended space the self-dual Yang-Mills or Einstein equations admit an interpretation as Cauchy-Riemann conditions associated with some harmonic analyticities, in a close suggestive analogy with $\mathcal{N}=2$ matter, SYM and SG theories in HSS as described above. These striking affinities between the implications of the harmonic methods in the extended supersymmetries and in the purely bosonic problems were e.g., used in [38,69] to construct unconstrained geometric formulations of the hyper-Kähler and quaternion-Kähler geometries. These formulations justified the interpretation of the general analytic q^+ interaction Lagrangian $L^{+4}(q^+, u)$ in (60) and its $\mathcal{N} = 2$ supergravity analog $\mathcal{L}^{+4}(Q^+, v^+, u^-)$ in (194) as the fundamental geometric quantities of both types of the complex Riemannian geometries.

HSS 方法与著名的扭量理论联系紧密。二者的共同点是通过二维球面 S^2 拓展了 (扭量理论中的) 时空与 (调和超空间方法中的) 超空间。在这类拓展空间中，自对偶杨-米尔斯方程或爱因斯坦方程可以被诠释为与某种调和解析性关联的柯西-黎曼条件，和上文所述 HSS 框架下的 $\mathcal{N} = 2$ 物质、SYM 与超引力理论有极具启发性的紧密类比。调和方法在拓展超对称与纯玻色子问题中的这些显著相似性，已被例如文献 [38, 69] 用于构造超凯勒与四元数-凯勒几何的无约束几何表述。这些表述印证了以下诠释:(60) 式中的广义解析 q^+ 相互作用拉格朗日量 $L^{+4}(q^+, u)$，以及它在 $\mathcal{N} = 2$ 超引力中的对应物 (194) 式中的 $\mathcal{L}^{+4}(Q^+, v^+, u^-)$，是两类复黎曼几何的基础几何量。

Finally, let us sketch several directions in which the HSS method in applications to SG models could be further developed.

最后，我们简要梳理可进一步发展 HSS 方法在超引力模型中应用的几个方向。

The formulation of $\mathcal{N} = 2$ SG theories and their couplings to matter through unconstrained analytic superfields with a nice geometric meaning opens a few perspectives, which so far were not still investigated in full. First of all, it should allow for a self-consistent procedure of quantization, like it has been done for $\mathcal{N} = 2$ SYM theory (see, e.g., reviews [73, 74]). In particular, the manifestly analytic superfield propagators and background superfield method could be constructed and used to study the structure of quantum corrections to the classical actions and the related geometric objects. Though $\mathcal{N} = 2$SG is certainly non-renormalizable by power-counting, the supersymmetry is capable to improve the UV behavior of $\mathcal{N} = 2$ SG models and it is just the off-shell supefield quantum SG machinery that would allow to check this in a manifestly supersymmetric and gauge invariant fashion.

通过具有清晰几何意义的无约束解析超场来构造 $\mathcal{N} = 2$ 超引力理论及其与物质的耦合，开辟了一些迄今为止尚未得到充分研究的研究方向。首先，该方法应该可以给出一套自洽的量子化流程，就像已经为 $\mathcal{N} = 2$ 超对称杨-米尔斯理论完成的工作那样 (例如参见综述 [73, 74])。具体而言，我们可以构造出显式解析超场传播子和背景超场方法, 并用其研究经典作用量及相关几何对象的量子修正结构。尽管按幂次计数 $\mathcal{N} = 2$SG 肯定是不可重整的，但超对称性能够改善 $\mathcal{N} = 2$ 超引力模型的紫外行为，而脱壳超场量子超引力工具正是能以显式超对称且规范不变的方式验证这一点的关键。

The same concerns $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ SG models in six dimension, the HSS formulation of which was pioneered in [75] (6D HSS was defined in [30,76].). It seems interesting to hybridize these $6D$ HSS SG models with the quantization methods developed for their $6D$ SYM cousins (see a recent review [77]).

六维下的 $\mathcal{N} = (1, 0)$ 和 $\mathcal{N} = (1, 1)$ 超引力模型也是如此，HSS 表述对这类模型的开创性研究出自文献 [75](六维 HSS 的定义见文献 [30,76])。将这些 $6D$ HSS 超引力模型与为对应 $6D$ 超对称杨-米尔斯理论开发的量子化方法 (参见近期综述 [77]) 相结合，是一个很有意义的研究方向。

Another area where the HSS approach could be useful is the construction of higher-order superfield invariants of $\mathcal{N} = 2$SG , including, e.g., $\mathcal{N} = 2$ conformal SG extensions of the standard conformal invariant (square of Weyl tensor [66]). To know the precise structure of such higher-derivative SG invariants is of interest for exploring the relationships with string theory.

HSS 方法另一个可能有用的领域是构造 $\mathcal{N}=2$SG 的高阶超场不变量，例如包括标准共形不变量 (外尔张量平方 [66]) 的 $\mathcal{N}=2$ 共形超引力拓展。弄清这类高阶导数超引力不变量的精确结构，对探索其与弦理论的联系十分重要。

Interesting prospective applications of the HSS approach to self-dual super-gravities were started in one of the last papers co-authored by V.I. Ogievetsky, Ref. [78]. The closely related new area of recent uses of $\mathcal{N}=2, 4D$ HSS approach is the higher-spin business attracting a lot of attention for last years (see. e.g., [79] and Refs. therein). Surprisingly, it turned out that the geometrical HSS formulation of the simplest $\mathcal{N}=2$ Einstein SG presented in section " $\mathcal{N}=2$ Einstein Supergravity "From Scratch"" admits direct generalizations as the off-shell unconstrained analytic superfield formulations for $\mathcal{N}=2$ superextensions of the integer higher spin $\mathbf{s}\geq 3$ theories. At the free level, $\mathcal{N}=2$ multiplet with a higher spin $\mathbf{s}$ is described by a triad of harmonic analytic superfields [17]

HSS 方法在自对偶超引力中有趣的前沿应用，起始于 V.I. 奥吉耶夫斯基合作完成的最后几篇论文之一，即文献 [78]。近年来，$\mathcal{N}=2, 4D$ HSS 方法应用的一个密切相关的新兴领域是高自旋问题，该问题近些年来受到了广泛关注 (例如参见 [79] 及其中引用的文献)。令人惊讶的是，我们发现，“从零开始构建 $\mathcal{N}=2$ 爱因斯坦超引力”一节中给出的最简单的 $\mathcal{N}=2$ 爱因斯坦超引力的几何 HSS 表述，可以直接推广为 $\mathcal{N}=2$ 对整数高自旋 $\mathbf{s}\geq 3$ 理论的超扩展的脱壳无约束解析超场表述。在自由层面，带有高自旋 $\mathbf{s}$ 的 $\mathcal{N}=2$ 多重态由一组调和解析超场的三重态描述 [17]

$$h^{++\alpha(s-1)\dot\alpha(s-1)}(\zeta,u)\,,\,h^{++\alpha(s-2)\dot\alpha(s-2)}(\zeta,u)\,,\,h^{++\alpha(s-1)\dot\alpha(s-2)+}(\zeta,u)\ \text{(and c.c)},$$

where $\alpha(s) := (\alpha_1 \dots \alpha_s), \dot\alpha(s) := (\dot\alpha_1 \dots \dot\alpha_s)$. For $\mathbf{s}=2$ the superfield content of $\mathcal{N}=2$SG is recognized. In [80] the superfield cubic vertices of interaction of these higher-spin gauge superfields with the matter hypermultiplets were also constructed. Though these theories (and their actions) are still available only at the linearized level, there is a hope to extend them to the full nonlinear level, closely following the appropriate steps in the HSS formulation of $\mathcal{N}=2$ SG theories. In this respect, it is of high importance to construct higher-spin generalizations of $\mathcal{N}=2$ conformal supergravity in HSS formulation (at least, at the linearization level).

其中 $\alpha(s) := (\alpha_1 \dots \alpha_s), \dot\alpha(s) := (\dot\alpha_1 \dots \dot\alpha_s)$ 。对于 $\mathbf{s}=2$ ，我们已经明确了 $\mathcal{N}=2$SG 的超场内容。文献 [80] 还构造了这些高自旋规范超场与物质超多重子相互作用的超场三次顶点。尽管这些理论 (及其作用量) 目前还只在线性化层面得到，但我们有望通过沿用 $\mathcal{N}=2$ 超引力理论 HSS 表述中的对应步骤，将其推广到完整非线性层面。就此而言，在 HSS 框架下构造 $\mathcal{N}=2$ 共形超引力的高自旋推广 (至少在线性化层面) 是非常重要的。

As a challenge to the HSS approach, there remains the problem of constructing off-shell $\mathcal{N}=3$ supergravity in terms of the appropriate unconstrained harmonic potentials. While the nice off-shell formulation of $\mathcal{N}=3, 4D$ SYM theory (equivalent to $\mathcal{N}=4$ SYM on shell) in $\mathcal{N}=3$ HSS with infinite numbers of auxiliary fields is known for many years [11, 12] , it is still mystery whether some analogous formulation of $\mathcal{N}=3, 4D$ SG can be invented. The main question is as to what is the appropriate off-shell HSS carrier of $\mathcal{N}=3$ superconformal Weyl multiplet. Once this is understood, the $\mathcal{N}=3$SG could be obtained from the superconformal one via some compensating procedure making use of the proper number of off-shell $\mathcal{N}=3$ Maxwell multiplets in the background of $\mathcal{N}=3$ Weyl multiplet (For on-shell component formulations of $\mathcal{N}=3$SG and vector $\mathcal{N}=3$ multiplets see, e.g., a recent preprint [81] and Refs. therein.). It is not excepted that solving this ambitious problem will require some new geometrical ideas beyond those incorporated within the standard $\mathcal{N}=3$ HSS approach (see, e.g., discussion in [82]).

对于 HSS 方法而言，仍存在一项挑战性课题: 如何利用恰当的无约束调和势构造离壳 $\mathcal{N}=3$ 超引力。多年前人们就已经得到了具有无限多辅助场的 $\mathcal{N}=3$ HSS 框架下 $\mathcal{N}=3, 4D$ 超对称杨-米尔曼 (SYM) 理论的良好离壳表述 (它和 $\mathcal{N}=4$ SYM 理论在壳上等价) [11, 12]，但目前仍不清楚能否构造出 $\mathcal{N}=3, 4D$ 超引力 (SG) 的类似表述。核心问题在于，什么是承载 $\mathcal{N}=3$ 超共形外尔多重态的恰当离壳 HSS 结构。一旦明确这一点，就可以通过补偿程序从超共形外尔多重态得到 $\mathcal{N}=3$SG : 即在 $\mathcal{N}=3$ 外尔多重态的背景上引入恰当数量的离壳 $\mathcal{N}=3$ 麦克斯韦多重态 (关于 $\mathcal{N}=3$SG 和矢量 $\mathcal{N}=3$ 多重态的壳分量表述，例如可参见近期预印本 [81] 及其中引用的文献)。解决这个高难度问题很可能需要引入超出标准 $\mathcal{N}=3$ HSS 框架的新几何思想，相关讨论可见例如文献 [82]。

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